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Outliers in Life-Testing Distributions

by

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To Kathryn

ABSTRACT

Much work has been done on outliers in normal populations and recently in exponential populations. Here we extend the study to the examination of outliers in three competing life-testing distributions. Assuming the exchangeable model of random variables, we examine the concepts of outlier-prone and outlier-resistant families as they apply to the Gamma, Lognormal, and Weibull families of density functions. We examine also the detection of outliers for these distributions and the estimation of parameters in the presence of one or more spurious observations.

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TABLE OF CONTENTS

CHAPTER		PAGE
I.	OUTLIERS AND OUTLIER-PRONENESS	1
	1.1 What is an outlier?	2
	1.2 The outlier problem	3
	1.3 Models for discordancy	6
	1.4 The concept of outlier-proneness	9
II.	SURVEY OF THE LITERATURE	13
	2.1 Detection of outliers in exponential and normal families using the exchangeable model	14
	2.2 Estimation in the presence of outliers	16
	2.2.1 Estimation for the single parameter exponential distribution	16
	2.2.2 Estimation for the normal distribution	18
III.	THE GAMMA DISTRIBUTION	21
	3.1 Characteristics of the Gamma Distribution	22
	3.2 Outlier-proneness of the exchangeable model with the Gamma distribution	25
	3.3 Detection of outliers	36
	3.4 Estimation for Gamma parameters	38
	3.4.1 Standard Estimators	38
	3.4.2 Estimators suggested for use	41

TABLE OF CONTENTS

CHAPTER		PAGE
IV.	THE LOGNORMAL DISTRIBUTION	52
	4.1 Characteristics of the Lognormal distribution	53
	4.2 Outlier-proneness of the exchangeable model with the Lognormal distribution	58
	4.3 Detection of outliers	74
	4.4 Estimation for Lognormal Parameters	78
	4.4.1 Standard Estimators	78
	4.4.2 Estimators suggested for use	79
V.	THE WEIBULL DISTRIBUTION	85
	5.1 Characteristics of the Weibull Distribution	86
	5.2 Outlier-proneness of the Weibull and related models	90
	5.3 Detection of outliers	99
	5.4 Estimation for Weibull parameters	109
	5.4.1 Standard Estimators	109
	5.4.2 Estimators suggested for use	114
REFERENCES		121
APPENDIX I	The A-, W-, and S-Rules and Premium/Protection	129
APPENDIX II	Asymptotic approximation of $\Gamma(\nu+1)$	132
APPENDIX III	Evaluation of $\int_{-\infty}^{\infty} \phi(x)\Phi(\nu x)dx$	135
APPENDIX IV	Tables of $u(r;n,k^*)$ for Weibull	136
APPENDIX V	Evaluation of moments of $EV_I(\xi,b)$	185
APPENDIX VI	Computer programs	188

LIST OF SYMBOLS

NOTATION	DESCRIPTION	PAGE
n	sample size	6
F	distribution function	6
G	distribution function	6
m	number of outliers	6
τ	location parameter	6
θ	scale parameter	6
S_n	sample of size n	6
$f(x;\underline{\theta})$	probability density function of target population P	6
$f(x;\underline{\xi})$	probability density function of spurious population Q	7
$L(\underline{x};\underline{\theta},\underline{\xi})$	likelihood function for the exchangeable model	7
$N(\tau,\theta^2)$	normal distribution with mean τ and variance θ^2	7
$EXP(\theta)$	exponential distribution with mean θ	7
k^*	coefficient of spuriosity	
I	set of spurious observations	8
$\mathcal{J}$	set of all $\binom{n}{m}$ possible sets of spurious observations	8
$P(k,n\|F)$	$P\{x_{(n)}-x_{(n-1)}>k(x_{(n-1)}-x_{(1)})\|x_1 \sim F\}$	9
$\Pi_1(k,n\|\mathcal{F})$	$\sup_{F\in\mathcal{F}} P(k,n\|F)$	10
$C(\xi,\theta)$	Cauchy distribution centered at ξ, scale θ	12

$u(r;n,k^*)$	$P(X_{(r)}$ is spurious in a sample of size $n)$	14
$V(x)$	$dG(x)/dF(x)$	14
$T_{m,n}$	$\dfrac{\sum_{i=1}^{n-m} x_{(i)} + mx_{(n-m)}}{n-m+1}$	16
$\hat{\theta}_{m,n}$	$\dfrac{\sum_{i=1}^{n-m} x_{(i)}}{n-m}$	17
T_t	$\dfrac{\sum_{i=1}^{n-1} x_{(i)}}{n}$	17
$D_{k^*}(\underline{x})$	$\begin{cases} T_{0,n} & \text{if } x_{(n)} - x_{(n-1)} < k(x_{(n-1)} - x_{(1)}) \\ T_t & \text{otherwise} \end{cases}$	17
$T_{k^*}(\underline{x})$	$\begin{cases} T_{0,n} & \text{if } x_{(n)} \leq C\bar{x}\ ,\ C > 0 \\ T_t & \text{otherwise} \end{cases}$	17
$\hat{\theta}_i$	$\sum_{\substack{j=1 \\ j\neq i}}^{n} x_j/n$	18
$\hat{\theta}_{CK}$	$\sum_{i=1}^{n} \omega_i \hat{\theta}_i$, $\omega_i = \dfrac{2r_i}{n(n+1)}$, r_i = rank of x_i	18
$GAM(\theta,\eta,\tau)$	$f(x,\theta,\eta,\tau) = \dfrac{(x-\tau)^{\eta-1} e^{-(x-\tau)/\theta}}{\theta^{\eta}\Gamma(\eta)}$, $x > \tau$, $\theta,\ \eta > 0$, $-\infty < \tau < \infty$	22
η	shape parameter	22
χ^2_{ν} df	Chi-square density with ν degrees of freedom	24
$h(x)$	HF; hazard function $= \dfrac{f(x)}{1-F(x)}$	24
E	event that $x_{(n)}$ is a (k,n)-outlier	26
$P(E\|\eta,k^*)$	$P(k,n\|L)$	26
$I_r(\eta,k^*)$	$\int_E (n-1)! \prod_{i\neq r} f(x_{(i)};\eta)(f(x_{(r)};k^*\eta)dx_{(1)}\cdots dx_{(r)}$	27
$u(r;n,\eta,k^*)$	$P(X_{(r)}$ is the spurious observation)	27

$I_n'(\eta,k^*)$		28
$\vartheta(\nu,t)$		28
$\hat{\theta}_{MLE}$	$\bar{x}/\eta$ maximum likelihood estimator of scale	38
$\tilde{x}$	$(\prod_{i=1}^{n} x_i)^{1/n}$	38
$\phi(z)$	Euler's psi function $\frac{d}{dz} \ln \Gamma(z)$	38
γ	Euler's constant .5772157	38
$\hat{\eta}_{MLE}$	Maximum likelihood estimator of shape	38
$\hat{\eta}_M$	Moment estimator of shape	39
$\hat{\theta}_M$	Moment estimator of scale	39
$\hat{\hat{\eta}}$	Lilliefors' estimator of shape η	39
$\hat{\hat{\theta}}$	Lilliefors' estimator of scale θ	39
η^*	Thom's estimator of shape η	40
θ^*	Thom's estimator of scale θ	40
M	$\ln(\bar{x}/\tilde{x}) = - \ln S_1$	40
$L(\underline{x};\theta,\eta,k^*,I)$		42
$\Lambda(\mu,\sigma)$	lognormal density $f(x;\mu,\sigma) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\{-\frac{1}{2}\frac{(\ln x-\mu)^2}{\sigma^2}\}$	53
$L(\underline{x};\sigma,k^*)$		59
$\phi(z)$	standard normal p.d.f.	71
$\Phi(z)$	standard normal distribution function at z	71
$u(r;n,\sigma,k^*)$	$P\{X_{(r)}$ is the spurious observation$\}$	75
$\hat{\mu}$	M.L.E., B.L.I.E. of μ, ; $\hat{\mu} = \bar{w} = \frac{\sum_{i=1}^{n} \ln x_i}{n}$	78
$\hat{\sigma}^2$	M.L.E. of σ^2	78
$\tilde{\sigma}^2$	B.L.I.E. of σ^2	78

$L(\underline{x};\mu,\mu_1,\sigma,I)$		81
$\hat{\mu}_{het}$	M.L.E. of μ	82
$\hat{\mu}_{1_{het}}$	M.L.E. of μ_1 under exchangeable model	82
$\hat{\sigma}^2_{het}$	M.L.E. of σ^2	82
$WEI(\theta,\eta,\tau)$	Weibull distribution $f(x;\theta,\eta,\tau) = \frac{\eta(x-\tau)^{\eta-1}}{\theta^{\eta}} \exp\{-(\frac{x-\tau}{\theta})^{\eta}\}$	86
$EV_I(\xi,b)$	Extreme-Value distribution $F(y) = 1 - \exp\{-\frac{\exp(y-\xi)}{b}\}$	88
$Q(k,n\|\eta)$		91
$L(\underline{x};\eta,k^*)$		92
$u(r;n,k^*)$		99
$L(u,p)$	Laplace transform of $e^{-\nu^u}$, Re $u > 0$	100
$\hat{\eta}$	M.L.E. of η	109
$\hat{\theta}$	M.L.E. of θ	109
$\tilde{b}_{BLIE}$	Mann and Fertig's (1973) BLIE of b	109
$\hat{b}_k$	Mann and Fertig's adaptation of Hassanein's estimator of b	110
b^*	Unbiased form of $\hat{b}_k$	110
b^*_{BLIE}	BLIE based on b^*	110
r	size of censored sample	110
$\hat{b}_B$	Bain's estimator of b	110
$k_{r,n}$		110
$\hat{\eta}_B$	Bain's estimator of η	111
$\hat{b}_s$	Englehardt and Bain's (1973) estimator of b	111
$\tilde{b}_{MF}$	Mann and Fertig's (1975) BLIE adaptation of $\hat{b}_s$	111
$\ell_{k,n}$		112

$\hat{b}_{EB}$	Englehardt and Bain's (1977) estimator of b	112
k_n		112
b^*_{EB}	BLIE based on $\hat{b}_{EB}$	113
$\hat{b}_{MN}$	Menon's estimator of b	113
$\hat{\xi}_{MN}$	Menon's estimator of ξ	113
$\hat{\eta}_{MN}$	Menon's estimator of η	113
$\hat{\theta}_{MN}$	Menon's estimator of θ	113
$\hat{\eta}_{D}$	Dubey's estimator of η	113
$\hat{b}_{MS}$	Murthy-Swartz estimator of b	114
$\hat{b}_{RA}$		117
$\hat{b}_{RW}$		117
$\hat{b}_{RS}$		117
$\hat{\xi}_{RA}$		117
$\hat{\xi}_{RW}$		117
$\hat{\xi}_{RS}$		117
$\hat{\mu}_A$	A-rule estimator for mean of normal distribution	129
$\hat{\sigma}^2_A$	A-rule estimator for variance of normal distribution	129
$\hat{\mu}_W$	W-rule estimator for mean of normal distribution	130
$\hat{\mu}_S$	S-rule estimator for mean of normal distribution	130
$\hat{\sigma}^2_W$	W-rule estimator for variance of normal distribution	130
$\hat{\sigma}^2_S$	S-rule estimator for variance of normal distribution	130
$\zeta(x)$	Reimann's zeta function	185

LIST OF TABLES

TABLE	DESCRIPTION	PAGE
Table 2.2.1	Comparison of $\hat{\theta}_{CK}$ with $T_{o,n}$, $T_{m,n}$ and $\hat{\theta}_{1,n}$	18
Table 2.2.2	Comparison of $\hat{\mu}_A$, $\hat{\mu}_S$, $\hat{\mu}_W$ for two choices of H_1	20
Table 3.4.1	Lilliefors estimator of η and $\frac{1}{\theta}$ based on M.L.E. and method of moments	39
Table 4.1.1	Properties of the Lognormal distribution	53
Table 5.3.1	Number of terms required to calculate $u(1;n,k^*)$ accurate to third decimal	101
Table 5.4.1	Optimal order statistics for the Murthy-Swartz estimator of b	114
Table 5.4.2	Computation of means, variances and Winsorized means and variances for 25 samples based on $EV_I(0, \frac{1}{\eta})$ with one spurious observation present.	118

LIST OF FIGURES

FIGURE	DESCRIPTION	PAGE
Fig. 3.1.1	The Gamma p.d.f.	23
Fig. 4.1.1	The Lognormal p.d.f.	55
Fig. 5.1.1	The Weibull p.d.f.	87

CHAPTER I

OUTLIERS AND OUTLIER-PRONENESS

What is an outlier and what is the outlier problem? To quote Ferguson (1961):

> In a sample of moderate size taken from a certain population it appears that one or two values are surprisingly far way from the main group. The experimenter is tempted to throw away the apparently erroneous values, and not because he is certain that the values are spurious. ... It is rather because he feels that other explanations are more plausible, and that the loss in the accuracy of the experiment caused by throwing away a couple of good values is small compared to the loss caused by keeping even one bad value.

Barnett and Lewis (1978) indicate that in the light of developments in outlier methodology over the last 15 years, Ferguson's formulation may be unduly restrictive. Outlying values need not be "bad" or "erroneous"; in fact they may be welcomed as indicating a useful treatment or a strain that was unexpectedly good.

1.1 What is an outlier?

What do we mean by an "outlier"? We shall use Barnett and Lewis' definition (1978): "an outlier in a set of data (is) ... an observation (or subset of observations) which appears to be inconsistent with the remainder of that set of data." We use the term "outlier" to characterize an observation that stands out in contrast to the rest of the data and is not consistent with what our mind feels constitutes a reasonable data set nor with our initial view of an appropriate probability model to describe the generation of our data. An observation is "spurious" if it is statistically unreasonable on the basis of some prescribed probability model. This would include an observation known to be generated by a different probability model; however a spurious observation will not necessarily show up as an outlier.

1.2 The outlier problem

Experimental scientists and others who deal with data are forced to make decisions about outliers - whether or not to include them in analysis, whether to make allowances for them on some compromise basis, etc. What worries an experimenter is whether or not some observations are genuine members of the main population. If these observations appear in the midst of the data they may not be conspicuous and are unlikely to distort inferences seriously. However, should the outlier appear extreme, it could create problems in attempts to represent the population and it could contaminate estimates and tests of population parameters.

Once we decide that outliers exist in a data set, we must decide on how to react to them. Methods to support outright rejection or to adjust values, prior to processing the principal mass of data will be related to any postulated model for that population. We are concerned with whether the extreme values are so extreme as to be unreasonable under our original model. This may indicate the data contains a "mistake" (that should be rejected or corrected) or it may indicate an alternative model for which the complete data set appears as a homogeneous sample. The outlier may be a 'foreign' random influence in an otherwise homogeneous data set - interesting to study in itself or only serving to obstruct analysis of the main data mass.

An assessment of the discordancy of some outliers is just the first stage of the study of outliers. Once an outlier is judged discordant we may:

1) decide to reject (or correct) it and analyse the remaining (modified) data on the original model,

2) decide to modify the model to incorporate the outliers in a non-discordant manner,

3) refine the analysis of the entire data set to accommodate outliers (i.e. render the analysis relatively impervious to their presence),

4) focus attention on the discordant outliers because they identify factors of unexpected value.

As examples of each of these situations, consider the following:

Fifteen residuals (about a simple model) of observations of the vertical semi-diameter of Venus, in seconds, made by Lt. Herndon in 1846 were

-0.30	+0.06	-0.05
-0.44	+0.63	+0.20
+1.01	-0.13	+0.18
+0.48	-1.40	+0.39
-0.24	-0.22	+0.10

Chauvenet (1863) declared -1.40 and +1.01 outliers. We may choose to reject them as "gross errors"; incorporate them by changing our model to a non-normal one or; accommodate them by using a robust estimation technique that employs Winsorization or an α-trimmed sample.

Consider also the data described by Pearson and Pearson (1931) on the capacity (in c.c.) of a sample of 17 male Moriori skulls.

1230	1318	1380	1420	1630	1378
1348	1380	1470	1445	1360	1410
1540	1260	1364	1410	1545	

The observation 1630 is suspicious and as such may be rejected as a measurement/recording error; or we may incorporate it in a non-discordant way by assuming a non-normal model (since biological data often require skew distributions); or identification of the outlier may reflect the presence of a small number of another species in the population being studied.

1.3 Models for discordancy

There are several possible models for discordancy.

i) Deterministic: this covers the cases of outliers caused by obvious identifiable gross measurement errors, etc. If the data set $\{x_i\}_{i=1}^n$ contains one observation, x_1, clearly resulting from measurement/recording error, we reject H: all $x_i \in F$ in favor of H_1: all $x_j \in F$ $(j \neq i)$ and x_i is different (requiring rejection or correction).

ii) Inherent: here we reject H: all observations are from F in favor of H_1: all observations are from G, where G has more or different inherent variability than F.

iii) Mixture: this model allows contamination of the sample by a few members of a population other than F. We reject H: all $x_i \in F$ in favor of H_1: all $x_i \in pF+(1-p)G$. Thus outliers reflect probability $1-p$ that observations arise from G.

iv) Slippage: here H_1 states all observations apart from some small number m arise independently from the initial model F indexed by location and scale parameters τ and θ, while the remaining m are independent observations from a modified version of F in which τ or θ is shifted. Which observation(s) come(s) from the shifted distribution is not specified. In most published work, F represents the normal distribution.

v) Exchangeable: this is an extension of the slippage alternative. A set of n observations $S_n = \{x_1,\ldots,x_n\}$ ideally comes from a target population P with probability density function (p.d.f.) $f(x;\underline{\theta})$. However the suspicion is that one of the observations, say x_i, is not

from the target population P but from a different population Q with p.d.f. $f(x;\underline{\xi})$. Prior to the experiment, there is no way to identify the possible (at most one spurious observation). Hence, a priori, each observation is equally likely to be the discordant one. Thus the random variables $X_1,\ldots,X_n$ are not independent but are exchangeable. For one outlier, the likelihood of the sample is given by

$$L(\underline{x}|\underline{\theta},\underline{\xi}) = \frac{1}{n}\sum_{i=1}^{n} f(x_i;\underline{\xi}) \prod_{j\neq i} f(x_j;\underline{\theta}) .$$

Definition 1.3.1: Let $\mathcal{F}$ be a family of absolutely continuous univariate distribution functions $\{F(x;\underline{\theta})\}$ indexed by a parameter $\underline{\theta} \in \Theta$. Then the random variables $X_1,\ldots,X_n$ are said to be exchangeable random variables based on $\mathcal{F}$ if their joint p.d.f. is of the form

$$\frac{1}{n}\sum_{r=1}^{n} \prod_{i\neq r} F'(x_i;\underline{\theta}_1)F'(x_r;\underline{\theta}_2),\ \underline{\theta}_1,\underline{\theta}_2 \in \Theta$$

where $F'(x;\underline{\theta}) = \frac{\partial}{\partial x} F(x;\underline{\theta})$.

Anscombe (1960) used this model for $f(x;\underline{\theta})$ as $N(\tau,\theta^2)$ and $f(x;\underline{\xi})$ as $N(\tau+a\theta,\theta^2)$. Guttman and Smith (1969,1971) and Guttman (1973a) used it for $f(x;\underline{\theta})$ as $N(\tau,\theta^2)$ and $f(x;\underline{\xi})$ as $N(\tau+a,\theta^2)$. Kale and Sinha (1971), Joshi (1972b), Kale (1974), Chikkagoudar and Kunchur (1980) and Rauhut (1982) have used it for $f(x;\underline{\theta})$ as $EXP(\theta)$ and $f(x;\underline{\xi})$ as $EXP(\theta/k^*)$, $0 < k^* \leq 1$.

For m outliers, we assume $x_{i_1},x_{i_2},\ldots,x_{i_{n-m}}$ come from a target

population P with p.d.f. $f(x;\underline{\theta})$, while $x_{i_{n-m+1}},\ldots,x_{i_n}$ come from populations $Q_1,Q_2,\ldots,Q_m$ with p.d.f. $f(x;\underline{\xi}_1),\ldots,f(x;\underline{\xi}_m)$. Some or all of the $\underline{\xi}_i$, $i = 1,\ldots,m$ may be identical (i.e. $\underline{\xi}_i = \underline{\xi}$). The association of the different observations with the different distributions is assumed to occur at random. For the case of all $\underline{\xi}_i = \underline{\xi}$, $i = 1,\ldots,m$, the likelihood of the sample is

$$L(\underline{x}|\underline{\theta},\underline{\xi}) = \frac{1}{\binom{n}{m}} \sum_{I\in\mathcal{J}} \prod_{i\in I} f(x_i;\underline{\xi}) \prod_{j\notin I} f(x_j;\underline{\theta})$$

where $I = (i_1,i_2,\ldots,i_n)$ is a selection of m integers from $\{1,2,\ldots,n\}$ and $\mathcal{J}$ is the set of all $\binom{n}{m}$ possible such choices. Certain relationships must exist between $f(x;\underline{\theta})$ and $f(x;\underline{\xi})$ for it to be reasonable that the suitability of the model will be reflected in outliers. For the case of one outlier, we need the discordant observation from Q to show up at one of the sample extremes. The exchangeable model also assumes that the maximum number of possible outliers is known.

1.4 The concept of outlier-proneness

Related to model development, though not actually a model to describe the occurrence of outliers, is a concept introduced by Neyman and Scott (1971) and furthered by Green (1974,1976) and Kale (1975b, 1975c, 1976). These papers considered a method of distinguishing between families of distributions by examining the extent to which they are liable to exhibit outliers.

Let S_n be a sample of $n \geq 3$ observations and let $x_{(1)},\cdots,x_{(n)}$ be the order statistics for this sample.

Definition 1.4.1: For a positive number k, $x_{(n)} \in S_n$ is a k-outlier on the right if $x_{(n)} - x_{(n-1)} > k\{x_{(n-1)}-x_{(1)}\}$. The definition of a k-outlier on the left is analogous (i.e $x_{(1)} \in S_n$ is a k-outlier on the left if $x_{(2)} - x_{(1)} > k\{x_{(n)}-x_{(2)}\}$.

Let $P(k,n|F)$ denote the probability that a sample of n observations from a distribution F will contain a k-outlier on the right.

Then, for jointly distributed random variables

$$P(k,n|F) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_{(n-1)}}\int_{(k+1)x_{(n-1)}-kx_{(1)}}^{\infty} g(x_{(1)},x_{(n-1)},x_{(n)})\, dx_{(n)}dx_{(1)}dx_{(n-1)}$$

where $g(x_{(1)},x_{(n-1)},x_{(n)})$ is the marginal joint p.d.f. of $X_{(1)}$, $X_{(n-1)}$, $X_{(n)}$ given by

$$g(x_{(1)},x_{(n-1)},x_{(n)}) = \int_{x_{(1)}}^{x_{(n-1)}} \cdots \int_{x_{(1)}}^{x_{(3)}} f(y_1,\ldots,y_n)dy_2\cdots dy_{n-2}$$

where f is the joint p.d.f. of $X_1,\ldots,X_n$.

For the special case of i.i.d. random variables we would have

$$P(k,n|F) = n(n-1)\int_{-\infty}^{\infty} dF(x)\int_{x}^{\infty} \{F(\frac{kx+y}{k+1}) - F(x)\}^{n-2} dF(y).$$

Let $\mathcal{F}$ be the family of distributions and let $\Pi_1(k,n|\mathcal{F})$ be the least upper bound of probabilities $P(k,n|F)$ for $F \in \mathcal{F}$.

Definition 1.4.2: A family $\mathcal{F}$ of distributions is (k,n)-outlier-prone on the right if $\Pi_1(k,n|\mathcal{F}) = 1$.

Definition 1.4.3: A family $\mathcal{F}$ of distributions is (k,n)-outlier-resistant on the right if $\Pi_1(k,n|\mathcal{F}) < 1$.

Definition 1.4.4: If a family $\mathcal{F}$ of distributions is (k,n)-outlier-prone on the right $\forall k > 0$, $\forall n > 2$ it is outlier-prone completely on the right (o.p.c.r.).

Definition 1.4.5: If a family $\mathcal{F}$ of distributions is (k,n)-outlier-resistant on the right $\forall k > 0$, $\forall n > 2$ it is outlier-resistant completely on the right. (o.r.c.r.).

Theorem 1.4.6: (Green (1974))

If $\mathcal{F}$ is a family of distributions and S_n is a random sample of n i.i.d. observations from $F \in \mathcal{F}$, then $\mathcal{F}$ is outlier-prone completely on the right (o.p.c.r.) iff $\mathcal{F}$ is (k,n)-outlier-prone on the right for some $k > 0$, $n > 2$.

For i.i.d. observations, Theorem 1.4.6 shows that it is not necessary to distinguish between the concepts of "(k,n)-outlier-prone on the right" and "outlier-prone completely on the right". This leads to the following definitions and theorem for cases in which the observations are i.i.d.

Definition 1.4.7: With respect to a random sample of n i.i.d. observations a family $\mathcal{F}$ is outlier-prone on the right (o.p.r.) iff it is (k,n)-outlier-prone on the right for some $k > 0$ and $n > 2$, or, equivalently, iff it is outlier-prone completely on the right.

Definition 1.4.8: With respect to a random sample of n i.i.d. observations a family $\mathcal{F}$ is outlier-resistant on the right (o.r.r.) iff it is not o.p.r.

Theorem 1.4.9: With respect to a random sample of n i.i.d. observations a family $\mathcal{F}$ is o.r.r. iff $\Pi_1(k,n|\mathcal{F}) < 1$ for some $k > 0$, $n > 2$, $F \in \mathcal{F}$.

Proof: i) Assume that family $\mathcal{F}$ is o.r.r. . Then $\mathcal{F}$ is not o.p.r. .

Therefore $\Pi_1(k,n|\mathcal{F}) \neq 1 \quad \forall k > 0, \forall n > 2.$

i.e. There exists at least one $k' > 0$, $n' > 2$, $F \in \mathcal{F}$ ∋ .

$$\sup_{F \in \mathcal{F}} P(k',n'|F) < 1 .$$

ii) Assume $\Pi_1(k,n|\mathcal{F}) < 1$ for some $k > 0$, $n > 2$, $F \in \mathcal{F}$. Then

$\Pi_1(k,n|\mathcal{F}) \neq 1 \quad \forall k > 0, \quad \forall n > 2 .$

Therefore $\mathcal{F}$ is not o.p.r.

From Definition 1.4.8 it follows that $\mathcal{F}$ is o.r.r. .

We will use the same definition of a (k,n)-outlier on the right when S_n represents n observations from the exchangeable model.

The implication is that for an outlier resistant family we are justified in seeking out and eliminating outliers. On the other hand, outlier-prone families should be used to model cases where outliers are common and in these cases we should seek ways of accommodating outliers.

Neyman and Scott (1971) showed that in general families differing only in location or scale parameters are outlier-resistant. Thus we may limit studies to subfamilies with standard values for location and scale. Both the family $N(\tau,\theta^2)$ or normal distributions (with mean τ and variance θ^2) and the family $C(\xi,\theta)$ of Cauchy distributions (centered at ξ and having scale parameter θ) are outlier-resistant.

CHAPTER II

SURVEY OF THE LITERATURE

Most studies of the outlier-problem assume an initial distribution that is either exponential or normal.

2.1 Detection of outliers in exponential and normal families using the exchangeable model.

A semi-Bayesian approach was used by Kale (1969), Kale and Sinha (1971), Chikkagoudar and Kunchur (1980), and Rauhut (1982) with respect to the one-parameter exponential family. It was assumed that $X_1,\ldots,X_n$ were such that $n-1$ observations were from $f(x;\theta_1) = \frac{1}{\theta_1}\exp\{\frac{-x}{\theta_1}\}$, $x \geq 0$, $\theta_1 > 0$ while one of the X_i's was distributed as $f(x;\theta_2)$, $\theta_2 \geq \theta_1$ (i.e. $\theta_2 = \theta_1/k^*$, $0<k^*\leq 1$). *A priori*, each X_i was equally likely to be distributed as $f(x;\theta_2)$. Kale and Sinha (1971) calculated $u(r;n,k^*)$, the probability that $X_{(r)}$, the r^{th} order statistic, corresponds to the spurious observation distributed as $f(x;\theta_2)$. It was shown that

$$u(r;n,k^*) = \frac{k^*\Gamma(n)\Gamma(n-r+k^*)}{\Gamma(n+k^*)\Gamma(n-r+1)}$$

was monotone increasing in r and hence $X_{(n)}$ had maximum posterior probability of being an outlier. Mount and Kale (1973) generalized this result for the case of $n-1$ observations with distribution function F and one with distribution function G where F and G are stochastically ordered ($G<F$), where *a priori* each observation is equally likely to be the spurious one, and where $\Psi(x) = \frac{dG}{dF}$ is monotone increasing in r. Kale (1974a) generalized these results for the case of m (≥ 1) possible outliers, where *a priori* each group of m observations $(X_{i_1},X_{i_2},\ldots,X_{i_m})$ is equally likely to come from distribution function G. Then $(x_{(n-m+1)},\ldots,x_{(n)})$ has maximum

posterior probability of corresponding to the set of spurious observations, provided Ψ is monotone increasing. Kale restricted the distributions to the single-parameter exponential family. In another paper, Kale (1974b) gave a completely Bayesian approach where $n-m$ observations were distributed as $f(x;\theta)$, m_1 observations were from $f(x;\theta_j)$, $j = 1,2,\ldots,m_1$, $\theta_j \leq \theta$, and m_2 observations were from $f(x;\theta_\ell)$, $\ell = 1,2,\ldots,m_2$, $\theta_\ell \geq \theta$, $m = m_1+m_2$ and f belonged to the single-parameter exponential family. Under the exchangeable model, it was shown that $\{(x_{(1)},\ldots,x_{(m)}),(x_{(n-m+1)},\ldots,x_{(n)})\}$ had maximum posterior probability of being the set of spurious observations.

For the case of the normal family of distributions, a completely Bayesian approach has been used by Box and Tiao (1968) where $f(x;\underline{\theta})$ is $N(\tau,\theta^2)$ and $f(x;\underline{\theta}_j)$ is $N(\tau,k^*\theta^2)$, $k^* \geq 1$ (Model A) and by Dempster and Rosner (1971) where $f(x;\underline{\theta})$ is $N(\tau,\theta^2)$ and $f(x;\underline{\theta}_j)$ is $N(\tau_j,\theta^2)$, $j = 1,2,\ldots,m$, $\tau_j \geq \tau$ (Model B). For $m = 1$, Model B has been handled as a slippage test by Ferguson (1961), Kudo (1956) and Paulson (1952).

2.2 Estimation in the presence of outliers.

Anscombe (1960) suggested a premium-protection approach (see Appendix 1) to estimation that has subsequently been used by Kale and Sinha (1971), Joshi (1972b), Chikkagoudar and Kunchur (1980), and Rauhut (1982) for estimation of the mean in the single-parameter exponential distribution and by Guttman and Smith (1969, 1971, 1973a) and Veale and Huntsberger (1969) for estimation in the normal distribution.

2.2.1 Estimation for the single-parameter exponential distribution.

Under homogeneity, the optimal estimator of θ is $T_{o,n} = \frac{\sum_{i=1}^{n} x_i}{n+1}$. However, under the exchangeable model with one outlier,

$$MSE(T_{o,n}|k^*) = \theta^2\{\frac{1}{n+1} + \frac{2}{n+1}(\frac{1-k^*}{k^*})^2\} \to \infty \quad \text{as} \quad k^* \to 0.$$

Among restricted L-type estimators $T(\underline{\ell}) = \sum_{j=1}^{n-m} \ell_j x_{(j)}$ which ignore the largest m observations, Kale and Sinha (1971) and Veale and Kale (1972) advocated the one-sided Winsorized mean

$$T_{m,n} = \frac{\sum_{i=1}^{n-m} x_{(i)} + m x_{(n-m)}}{n-m+1}$$

for which $MSE(T(\underline{\ell})|k^*=1)$ is minimum and $MSE(T_{m,n}|k^*<1)$ shows a gain in efficiency relative to $T_{o,n}$ for k^* sufficiently small. Joshi (1972(b)) considered choice of m and showed for k^* small, substantial gains in relative efficiency are possible. Samples up to size $n = 20$ were considered.

Kale (1974a) considered maximum likelihood estimation (M.L.E.) for the exchangeable model with m possible upper outliers and obtained $\hat{\theta}_{m,n} = \frac{\sum_{i=1}^{n-m} x_{(i)}}{n-m}$, the trimmed mean, as MLE of θ. In comparing the trimmed mean $\hat{\theta}_{1,n} = \frac{\sum_{i=1}^{n-1} x_{(i)}}{n-1}$ with the Winsorized estimator $T_{1,n} = \frac{\sum_{i=1}^{n-1} x_{(i)} + x_{(n-1)}}{n}$ for the case of a single upper outlier, it was shown that

$$MSE(\hat{\theta}_{1,n} | k^*=1) > MSE(T_{1,n} | k^*=1)$$

but $\lim_{k^* \to 0} MSE(\hat{\theta}_{1,n} | k^*) = \frac{1}{n-1} < \lim_{k^* \to 0} MSE(T_{1,n} | k^*)$. Thus $\hat{\theta}_{1,n}$ provides more protection than $T_{1,n}$ but for a higher premium (See Appendix I).

Rauhut (1982) suggested two 'testimators":

$$D_{k^*}(\underline{x}) = \begin{cases} T_{o,n} & , \text{ if } x_{(n)} - x_{(n-1)} < k\{x_{(n-1)} - x_{(1)}\} \\ \frac{\sum_{i=1}^{n-1} x_{(i)}}{n} = T_t & , \text{ otherwise} \end{cases}$$

$$T_{k^*}(\underline{x}) = \begin{cases} T_{o,n} & , \text{ if } x_{(n)} < C\bar{x}, \quad C > 0 \\ T_t & , \text{ otherwise} \end{cases}$$

and showed $T_{k^*}(\underline{x})$ is preferred over $T_{o,n}$, T_t and D_{k^*} since the premium is small compared to the protection it provides.

Chikkagoudar and Kunchur (1980) proposed using $\hat{\theta}_{CK} = \sum_{i=1}^{n} w_i \hat{\theta}_i$

where $\hat{\theta}_i = \frac{\sum_{\substack{j=1 \\ j\neq i}}^{n} x_j}{n}$, $w_i = \frac{2r_i}{n(n+1)}$, r_i = rank of x_i in the complete sample. This estimator is more efficient than:

TABLE 2.2.1	Alternative Estimator	
	$T_{o,n}$	$T_{m,n}$	$\hat{\theta}_{1,n}$
When to use $\hat{\theta}_{CK}$	$n\geq 4, k^* \leq .75$	$n\geq 6$, $.40 \leq k^* \leq .75$	all n, $k^* \geq .45$
	$n=3, k^* \leq .70$	$n=4,5$, $.35 \leq k^* \leq .75$	
	$n=2, k^* \leq .65$	$n=3$, $.30 \leq k^* \leq .70$	
		$n=2$, $.35 < k^* < .65$	

and it is independent of k^*.

2.2.2 Estimation for the normal distribution

Kale (1974a) has shown that the method of maximum likelihood applied to the exchangeable model involving normal distributions with change in location leads to estimators that are trimmed means.

For the case of $n-1$ observations from $N(\tau, \theta^2)$ and one observation from $N(\tau+k^*\theta, \theta^2)$, $k^* \geq 0$, $\bar{x}$ is UMVUE, MLE for τ if $k^* = 0$ but $\bar{x}$ is biased and $MSE(\bar{x}) = \frac{\theta^2 k^{*2}}{n}$ if $k^* > 0$. Thus an alternative estimator might be $T(\underline{x})$ where $T(\underline{x})$ is one of the following:

$$T(\underline{x}) = \begin{cases} \dfrac{\sum_{i=1}^{n-1} x_{(i)}}{n} = T_t \\ \dfrac{1}{n}\left\{\sum_{i=1}^{n-1} x_{(i)} + x_{(n-1)}\right\} = T_{1,n} \\ A(\underline{x}) = \begin{cases} \bar{x} \quad , \text{ if } x_{(n)} - \bar{x} \leq C_\alpha S \\ \dfrac{\sum_{i=1}^{n-1} x_{(i)}}{n-1} \quad , \text{ otherwise} \end{cases} \end{cases}$$

The premium paid for using T instead of $\bar{x}$ is given by

$$\frac{1}{\theta^2} \text{MSE}(T|k^*=0) - \frac{1}{\theta^2} \text{MSE}(\bar{x}|k^*=0).$$

The protection obtained by using T instead of $\bar{x}$ is given by

$$\frac{1}{\theta^2} \text{MSE}(\bar{x}|k^*>0) - \frac{1}{\theta^2} \text{MSE}(T|k^*>0).$$

A detailed study of the accommodation of outliers in slippage models appears in Guttman and Smith (1969, 1971) and Guttman (1973a), using premium-protection. They consider three methods for estimating the mean $\mu = \tau$:

i) modified trimming $\hat{\mu}_A$ (A-rule)

ii) modified winsorization $\hat{\mu}_W$ (W-rule)

iii) semi-winsorization $\hat{\mu}_S$ (S-rule)

(see Appendix I).

Table 2.2.2. Comparison of $\hat{\mu}_A, \hat{\mu}_S$ and $\hat{\mu}_W$

Form of H_1 (one or two outliers)

	Location-slippage $N(\tau+k^*, \theta^2), k^*>0$	Scale-slippage $N(\tau, \theta^2 k^*), k^*>1$
$\hat{\mu}_A$	Best for large k^*	Not a contender
$\hat{\mu}_W$	Best for intermediate k^*	Best for large k^*
$\hat{\mu}_S$	Best for small k^*	Best for small k^*

The results are based on sample sizes up to $n = 20$, $\theta^2 = \sigma^2$ known, and only for $n = 3$, when θ^2 is unknown.

Guttman and Smith (1971) defined dispersion estimators $\hat{\sigma}_A^2$, $\hat{\sigma}_W^2$ and $\hat{\sigma}_S^2$ of $\hat{\sigma}^2$ analogously to $\hat{\mu}_A$, $\hat{\mu}_W$ and $\hat{\mu}_S$ (see Appendix I); $\hat{\sigma}_W^2$ is not worth considering when $\mu = \tau$ is known, $\hat{\sigma}_W^2$ and $\hat{\sigma}_A^2$ are not worth considering when τ is unknown.

CHAPTER III

The Gamma Distribution

One of the most common life-testing distributions is the gamma distribution. We shall examine the exchangeable model based on the gamma distribution and show that, in the case of a shape change, it is outlier-prone completely. For changes in shape or scale parameter, we shall determine which observation is most likely to be the spurious one. We shall also consider estimation in the presence of an outlier.

3.1 Characteristics of the Gamma Distribution

Consider the three-parameter gamma distribution given by

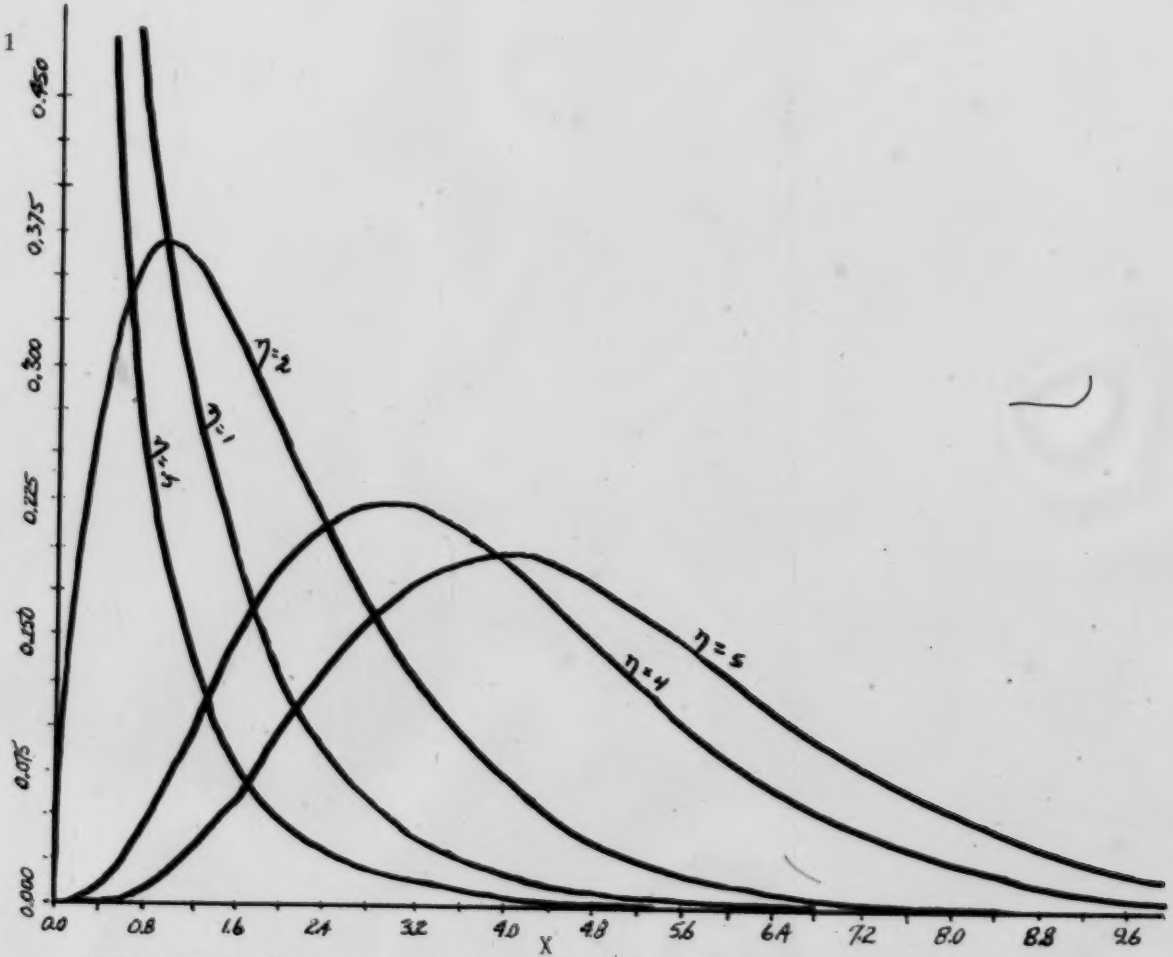
$$f(x;\theta,\eta,\tau) = \frac{(x-\tau)^{\eta-1} e^{-\frac{x-\tau}{\theta}}}{\theta^{\eta}\Gamma(\eta)}, \quad x > \tau,\ \theta,\eta > 0,\ -\infty < \tau < \infty.$$

We may denote this p.d.f. by $GAM(\theta,\eta,\tau)$. The shape parameter is η, the scale parameter θ, and the location parameter τ. For $\tau = 0$ or known, this reduces to the two-parameter gamma distribution where

$$f(x;\theta,\eta) = \frac{e^{-\frac{x}{\theta}} x^{\eta-1}}{\theta^{\eta}\Gamma(\eta)}, \quad x > 0,\ \eta,\theta > 0.$$

For $\eta = 1$ we have the exponential distribution with parameter θ and for integer η we have the Erlang distribution. For $\eta \leq 1$ and fixed θ, the p.d.f. is decreasing in x and unbounded ($\eta<1$) near the origin. The mean is $\theta\eta$, variance is $\theta^2\eta$, $E(X^r) = \frac{\theta^r\Gamma(\eta+r)}{\Gamma(\eta)}$ and the moment-generating function is $M_x(t) = (1-\theta t)^{-\eta}$, $t < \frac{1}{\theta}$. The shape parameter η is the reciprocal of the squared coefficient of variation.

Fig.3.1.1



Gamma p.d.f. for various values of η (θ =1, τ =0)

If T denotes the waiting time until the η^{th} occurrence in a Poisson process with intensity λ, then $T \sim GAM(\theta = \frac{1}{\lambda}, \eta, 0)$. The waiting time until the first occurrence may be denoted as $GAM(\theta = \frac{1}{\lambda}, 1, 0)$ which is $EXP(\theta = \frac{1}{\lambda})$. Thus the gamma distribution is a natural extension of the exponential distribution. The two-parameter exponential may be denoted as $GAM(\theta = \frac{1}{\lambda}, \eta = 1, \tau)$ where τ is a location parameter.

If $X_1, \ldots, X_n$ i.i.d. $GAM(\theta, m_i, 0)$ then $Y = \sum_{i=1}^{n} X_i \sim GAM(\theta, \sum_{i=1}^{n} m_i, 0)$. Consider the case of a component with $n-1$ spare parts. If X_i, $i = 1,2,\ldots,n$ denote the lifetimes of the component and the spares and if each is distributed $GAM(\theta,1,0) = EXP(\theta = \frac{1}{\lambda})$, then the lifetime of the system, assuming use of the $n-1$ spares, is $Y = \sum_{i=1}^{n} X_i \sim GAM(\theta,n,0)$ or $\frac{2Y}{\theta} \sim \chi^2_{2ndf}$. Note also that if $Y \sim GAM(2,\eta,0)$ then $Y \sim \chi^2_{2\eta\ df}$. The hazard function (HF) $h(x) = \lambda = \frac{1}{\theta}$ for $\eta = 1$; $h(x) \to \lambda^-$ as $x \to \infty$ for $\eta < 1$ and $h(0) = 0$; $h(x) \to \lambda^+$ as $x \to \infty$ and $h(0) = \infty$ for $\eta > 1$. Consequently the gamma distribution can model systems in a regular maintenance program where the failure rate is likely to increase initially but then stabilize.

Outliers in gamma samples arise in any context where Poisson processes are appropriate basic models, e.g. traffic flow, biological aggregation, failure of electronic equipment. They also occur in the context of a shifted exponential or gamma distribution. Outliers in χ^2 samples arise in ANOVA; outliers in gamma samples of arbitrary shape parameter arise with skew-distributed data, for which the gamma distribution is often a useful model.

3.2 Outlier-proneness of the exchangeable model with the Gamma distribution.

Neyman and Scott (1971) showed that for i.i.d.r.v.'s the family of gamma distributions indexed by shape parameter η is outlier-prone completely on the right. Let us consider now the exchangeable model.

CASE I: Scale change

Kale (1975b) showed that the exchangeable model with (at most) one spurious observation for scale parameter families for non-negative random variables involving a possible change in scale is outlier-prone completely. This would apply to the exchangeable model involving the gamma distribution with possible change in scale parameter θ.

CASE II: Shape change

Now consider the exchangeable model with (at most) one spurious observation based on the gamma distribution with possible change in the shape parameter η. Without loss of generality (w.l.o.g) we may take $\theta = 1$, $\tau = 0$.

Then $n-1$ observations have p.d.f. $f(x;\underline{\theta})$ which is $GAM(1,\eta,0)$ and one observation has p.d.f. $f(x;\underline{\xi})$ which is $GAM(1,k^*\eta,0)$ where $k^* \geq 1$. We shall show that this model is outlier-prone completely on the right.

Now we may write the likelihood as

$$L(x_1,\ldots,x_n;\eta,k^*) = \frac{1}{n}\sum_{r=1}^{n}\prod_{i\neq r} f(x_i;\eta)f(x_r;k^*\eta), \quad x_i \geq 0,\ k^* \geq 1,\ \eta > 0$$

($k^*\geq 1$ since we are considering $x_{(n)}$ as a possible outlier). Letting

E denote the event that $x_{(n)}$ is a (k,n)-outlier on the right and $P(k,n|L) = P(E|\eta,k^*)$, we must show that $\sup_{\substack{\eta>0 \\ k^*\geq 1}} P(E|\eta,k^*) = 1$ in order for this family to be (k,n)-outlier-prone.

The joint p.d.f. of the order statistics $x_{(1)},\ldots,x_{(n)}$ is given by

$$g(x_{(1)},\ldots,x_{(n)};\eta,k^*) = n!L(x_{(1)},\ldots,x_{(n)};\eta,k^*)$$

$$= (n-1)!\sum_{r=1}^{n}\prod_{i\neq r} f(x_{(i)};\eta)f(x_{(r)};k^*\eta),$$

$$0 < x_{(1)} < \ldots < x_{(n)} < \infty .$$

Thus

$$P(E|\eta,k^*) = \int_E g(x_{(1)},\ldots,x_{(n)};\eta,k^*)dx_{(1)}\cdots dx_{(n)}$$

$$= \sum_{r=1}^{n}\int_E (n-1)!\prod_{i\neq r} f(x_{(i)};\eta)f(x_{(r)};k^*\eta)dx_{(1)}\cdots dx_{(n)}$$

$$= \sum_{r=1}^{n} I_r(\eta,k^*)$$

and E is such that $0 < x_{(1)} < x_{(2)} < \ldots < x_{(n-1)} < \frac{kx_{(1)}+x_{(n)}}{k+1} < x_{(n)} < \infty$.

Lemma 3.2.1: $0 \leq I_r(\eta,k^*) \leq u(r;n,\eta,k^*)$ where $u(r;n,\eta,k^*) = P\{X_{(r)}$ is the "spurious" observation whose p.d.f. is $f(x,k^*\eta)\}$.

Proof:

$$u(r;n,\eta,k^*) = (n-1)!\int_S \prod_{i\neq r} f(x_{(i)};\eta)f(x_{(r)};k^*\eta)dx_{(1)},\ldots,dx_{(n)}$$ where

S is such that $0 < x_{(1)} < x_{(2)} < \ldots < x_{(n-1)} < x_{(n)} < \infty$.

$$\text{Now } I_r(\eta,k^*) = (n-1)!\int_E \prod_{i\neq r} f(x_{(i)};\eta)f(x_{(r)};k^*\eta)dx_{(1)},\ldots dx_{(n)}$$ where

E is such that $0 < x_{(1)} < x_{(2)} < \ldots < x_{(n-1)} < \frac{kx_{(1)}+x_{(n)}}{k+1} < x_{(n)} < \infty$.

Since $E \subset S$, $0 \leq I_r(\eta,k^*) \leq u(r;n,\eta,k^*)$, $r = 1,2,\ldots,n$.

Theorem 3.2.2: $\lim_{k^*\to\infty} u(r;n,\eta,k^*) = \begin{cases} 1, & r = n \\ 0, & r = 1,2,\ldots,n-1 \end{cases}$.

Proof: Now $u(r;n,\eta,k^*) = \binom{n-1}{r-1}\int_0^\infty \{F(y;\eta)\}^{r-1}\{1-F(y;\eta)\}^{n-r}f(y;k^*\eta)dy$

where $F(y;\eta) = \int_{-\infty}^{y} f(x;\eta)dx$.

Consider now

$$u(n;n,\eta,k^*) = \int_{y=0}^{\infty}\{\int_{x=0}^{y}\frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}dx\}^{n-1}\frac{e^{-y}y^{k^*\eta-1}}{\Gamma(k^*\eta)}dy,\ k^* \geq 1,\ \eta > 0\ .$$

(without loss of generality $\theta = 1$, $\tau = 0$)

Let

$$I_n'(\eta,k^*) = \int_{y=0}^{\infty} \{\int_0^y \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\}^{n-1}\, e^{-y}y^{\nu}dy \quad \text{for} \quad \nu = k^*\eta - 1.$$

Now setting $y = \nu t$, $dy = \nu dt$

$$I_n'(\eta,k^*) = \int_{t=0}^{\infty} \{\int_0^{\nu t} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\}^{n-1}\, e^{-\nu t}(\nu t)^{\nu}\nu dt$$

$$= \nu^{\nu+1} \int_{t=0}^{\infty} \vartheta(\nu,t) e^{\nu(\ell n\, t - t)} dt$$

where

$$\vartheta(\nu,t) = [\int_0^{\nu t} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx]^{n-1}.$$

and where there is no loss in generality in assuming $\nu > 0$ (i.e. $k^*\eta > 1$) since we are interested in $\lim_{k^*\to\infty} I_n'(\eta,k^*)$.

For $0 \le t \le \alpha$, $\vartheta(\nu,0) \le \vartheta(\nu,t) \le \vartheta(\nu,\alpha)$ and $0 \le \vartheta(\nu,t) \le 1$ for all ν,t and $\vartheta(\nu,t) \to 1$ a.e. in t as $\nu \to \infty$ (i.e. as $k^* \to \infty$). We may write $\int_0^{\infty} \vartheta(\nu,t) e^{\nu(\ell n\, t - t)} dt = \int_0^1 \vartheta(\nu,t) e^{\nu(\ell n\, t - t)} dt$

$$+ \int_1^{\infty} \vartheta(\nu,t) e^{\nu(\ell n\, t - t)} dt.$$

For each of the integrals on the right hand side, the dominant part occurs in the neighborhood of the point when $\ell n\, t - t$ is maximum, i.e. $t = 1$. Following Copson (1967, p. 36 ff), since $0 \le \vartheta(\nu,t) \le 1$ for all ν and t and $\vartheta(\nu,t)$ is continuous and increasing in t there

is a number $t_0 (0<t_0\leq 1)$ such that

$$\int_0^1 \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt = \vartheta(\nu,t_0)\int_0^1 e^{\nu(\ell n\ t-t)}dt\ ,\quad 0\leq t_0\leq 1\ .$$

We cannot have $t_0 = 0$ since $\vartheta(\nu,0) = 0$ while

$$\int_0^1 \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt > \int_{.5}^1 \vartheta(\nu,t)e^{\nu(\ell nt-t)}dt$$

$$> \frac{1}{2}\vartheta(\nu,\ \tfrac{1}{2})e^{-\nu(-\frac{1}{2}\ -\ell n\ 2)} > 0$$

and

$$\int_1^\infty \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt = \lim_{\alpha\to\infty}\int_1^\alpha \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt$$

$$= \lim_{\alpha\to\infty}\vartheta(\nu,t_1)\int_1^\alpha e^{\nu(\ell n\ t-t)}dt,\quad 1\leq t_1\leq\alpha.$$

Thus

$$\int_0^\infty \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt \sim \int_0^\infty e^{\nu(\ell n\ t-t)}dt$$

$$\sim 1\cdot\frac{\Gamma(\nu+1)}{\nu^{\nu+1}}\quad\text{as}\quad \nu\to\infty$$

(see Appendix II). As a result, we may write

$$I'_n(\eta,k^*) \sim \nu^{\nu+1}\cdot 1\cdot\frac{\Gamma(\nu+1)}{\nu^{\nu+1}} = \Gamma(\nu+1) \quad \text{as} \quad \nu \to \infty$$

and

$$u(n;n,\eta,k^*) = \frac{I'_n(\eta,k^*)}{\Gamma(k^*\eta)} \quad \text{where} \quad \nu = k^*\eta-1$$

$$\to 1 \quad \text{as} \quad \nu \to \infty \quad (\text{i.e. as} \quad k^* \to \infty).$$

For $u(r;n,\eta,k^*)$, $r = 1,2,\ldots,n-1$ we have

$$0 \le u(r;n,\eta,k^*) = \binom{n-1}{r-1}\int_{y=0}^{\infty} \left\{\int_0^y \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{r-1}$$

$$\left\{1 - \int_0^y \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{n-r} \frac{e^{-y}y^{k^*\eta-1}}{\Gamma(k^*\eta)}\,dy$$

$$= \binom{n-1}{r-1}\frac{I'_r(\eta,k^*)}{\Gamma(k^*\eta)}$$

where

$$I'_r(\eta,k^*) = \int_{t=0}^{\infty} \left\{\int_0^{\nu t} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{r-1}\left\{1 - \int_0^{\nu t} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\right\}^{n-r} e^{-\nu t}\nu^{\nu+1}t^{\nu}\,dt,$$

$\nu = k^*\eta-1$, $y = \nu t$, $dy = \nu dt$ and again we assume $\nu > 0$. Therefore

$$I'_r(\eta,k^*) = \nu^{\nu+1}\int_0^\infty \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt$$

$$= \nu^{\nu+1}\lim_{\alpha\to\infty}\int_0^\alpha \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt\ .$$

Now $\vartheta(\nu,t) = \{\int_0^{\nu t}\frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}dx\}^{n-1}\{1-\int_0^{\nu t}\frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}dx\}^{n-r} \le 1$ and as $k^*\to\infty$, $\nu\to\infty$ and $\vartheta(\nu,t)\to 0$ almost everywhere on $[0,\infty]$. Since $\vartheta(\nu,t)$ is continuous and bounded for all t

$$\int_0^\alpha \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt = \vartheta(\nu,t_0)\int_0^\alpha e^{\nu(\ell n\ t-t)}dt \qquad 0\le t_0\le\alpha$$

$$\le \vartheta(\nu,t_0)\int_0^\infty e^{\nu(\ell n\ t-t)}dt = \vartheta(\nu,t_0)\frac{\Gamma(\nu+1)}{\nu^{\nu+1}}\ .$$

Therefore $0 \le u(r;n,\eta,k^*) \le \frac{\binom{n-1}{r-1}}{\Gamma(k^*\eta)}\nu^{\nu+1}\lim_{\alpha\to\infty}\frac{\vartheta(\nu,t_0)\Gamma(\nu+1)}{\nu^{\nu+1}} = \binom{n-1}{r-1}\lim_{\alpha\to\infty}\vartheta(\nu,t_0)$. But $\binom{n-1}{r-1}\lim_{\alpha\to\infty}\vartheta(\nu,t_0)\to 0$ as $\nu\to\infty$ (i.e. as $k^*\to\infty$) and hence $u(r;n,\eta,k^*)\to 0$ as $\nu\to\infty$ for $r = 1,2,\ldots,n-1$.

Theorem 3.2.3: The exchangeable model with (at most) one spurious observation based on the gamma distribution with a change in shape parameter is outlier-prone completely on the right.

Proof: We need to show $\sup_{\substack{\eta>0\\ k^*\ge 1}} P(E|\eta,k^*) = 1$.

Since $P(E|\eta,k^*) = \sum_{r=1}^{n} I_r(\eta,k^*)$,

$$\lim_{k^*\to\infty} P(E|\eta,k^*) = \sum_{i=1}^{n} \lim_{k^*\to\infty} I_r(\eta,k^*).$$

From Lemma 3.2.1, $0 \leq I_r(\eta,k^*) \leq u(r;n,\eta,k^*)$, $r = 1,2,\ldots,n$.

By Theorem 3.2.2, $\lim_{k^*\to\infty} u(r;n,\eta,k^*) = \begin{cases} 0 & , r = 1,2,\ldots,n-1 \\ 1 & , r = n \end{cases}$.

Thus $\lim_{k^*\to\infty} I_r(\eta,k^*) = 0$, $r = 1,2,\ldots,n-1$.

and $\lim_{k^*\to\infty} P(E|\eta,k^*) = \lim_{k^*\to\infty} I_n(\eta,k^*)$.

Now we need only show that $\lim_{k^*\to\infty} I_n(\eta,k^*) = 1$ in order for this model to be outlier-prone completely on the right.

But

$$I_n(\eta,k^*) = (n-1)! \int_E \prod_{i\neq n} f(x_{(i)};\eta) f(x_{(n)};k^*\eta)\, dx_{(1)} \cdots dx_{(n)}$$

$$= (n-1)! \int_E \prod_{i\neq n} \frac{e^{-x_{(i)}} x_{(i)}^{\eta-1}}{\Gamma(\eta)} \cdot \frac{e^{-x_{(n)}} x_{(n)}^{k^*\eta-1}}{\Gamma(k^*\eta)}\, dx_{(1)} \cdots dx_{(n)}$$

where E is such that $0 < x_{(1)} < x_{(2)} < \cdots < x_{(n-1)} < \frac{kx_{(1)}+x_{(n)}}{k+1} < x_{(n)} < \infty$.

Let $y_1,\ldots,y_n$ denote the order statistics $x_{(1)},\ldots,x_{(n)}$, respectively. Then we may write

$$I_n(\eta,k^*) = (n-1) \iint_{0<y_1<y_n<\infty} \{\int_0^{\frac{ky_1+y_n}{k+1}} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx - \int_0^{y_1} \frac{e^{-x}x^{\eta-1}}{\Gamma(\eta)}\,dx\}^{n-2}$$

$$\frac{e^{-y_1}y_1^{\eta-1}}{\Gamma(\eta)} \frac{e^{-y_n}y_n^{k^*\eta-1}}{\Gamma(k^*\eta)}\,dy_1dy_n$$

$$= \int_{y_n=0}^{\infty} [\int_{y_1=0}^{y_n} (n-1)\{F(\frac{ky_1+y_n}{k+1};\eta)-F(y_1;\eta)\}^{n-2} f(y_1;\eta)dy_1]$$

$$f(y_n;k^*\eta)dy_n$$

where $f(y;\eta) = \frac{e^{-y}y^{\eta-1}}{\Gamma(\eta)}$, $y > 0$. Then

$I_n(\eta,k^*) = \frac{1}{\Gamma(k^*\eta)} \int_{y_n=0}^{\infty} \vartheta(y_n)e^{-y_n}y_n^{k^*\eta-1}dy_n$ where

$\vartheta(y_n) = \int_{y_1=0}^{y_n} (n-1)\{F(\frac{ky_1+y_n}{k+1};\eta)-F(y_1;\eta)\}^{n-2} f(y_1;\eta)dy_1$. Setting

$y_n = \nu t$ where $\nu = k^*\eta-1$, we obtain

$$(3.2.1) \qquad I_n(\eta,k^*) = \frac{\nu^{\nu+1}}{\Gamma(\nu+1)} \int_{t=0}^{\infty} \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$$

$$= \frac{\nu^{\nu+1}}{\Gamma(\nu+1)} \lim_{\alpha\to\infty} \int_{t=0}^{\alpha} \vartheta(\nu,t)e^{\nu(\ln t-t)}dt$$

where $\vartheta(\nu,t) = \vartheta(\nu t) = \int_{y_1=0}^{\nu t} (n-1)\{F(\frac{ky_1+\nu t}{k+1};\eta)-F(y_1;\eta)\}^{n-2} f(y_1;\eta)dy_1$.

As $k^* \to \infty$, $\nu \to \infty$ and the major contribution to this integral in 3.2.1 occurs in the neighborhood of $t = 1$. Also $0 \leq \vartheta(\nu,t) \leq 1$ for all ν,t and $\vartheta(\nu,t) \to 1$ a.e. on $[0,\infty)$ as $\nu \to \infty$ (i.e. as $k^* \to \infty$), since

$$\lim_{\nu\to\infty} \left[F\left(\frac{ky_1+\nu t}{k+1};\eta\right)-F(y_1;\eta)\right] = 1-F(y_1;\eta) \text{ a.e. on } [0,\infty).$$

If $t = 0$, $\vartheta(\nu,t) = (n-1)\int_0^{\nu t} \left[F\left(\frac{ky_1+\nu t}{k+1};\eta\right)-F(y_1;\eta)\right]^{n-2} f(y_1;\eta)dy_1$ and $\vartheta(\nu,0) = 0$. But $\vartheta(\nu,t)$ is nonnegative, bounded, increasing and absolutely continuous in t. Thus $\exists t_0 \in [0,\infty) \ni . \int_0^{\alpha} \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt$ $= \vartheta(\nu,t_0) \int_0^{\alpha} e^{\nu(\ell n\ t-t)}dt$. On the other hand for any $0 < \xi < 1$

$$\int_0^{\alpha} \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt > \int_{\xi}^{1} \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt$$

$$> \vartheta(\nu,\xi)e^{\nu(\ell n\ \xi-\xi)}(1-\xi)$$

$$> 0.$$

Therefore $t_0 \neq 0$.

Thus $\int_0^{\alpha} \vartheta(\nu,t)e^{\nu(\ell n\ t-t)}dt \sim \int_0^{\infty} e^{\nu(\ell n\ t-t)}dt = \frac{\Gamma(\nu+1)}{\nu^{\nu+1}}$ as $\nu \to \infty$

and

$$I_n(\eta,k^*) = \frac{\nu^{\nu+1}}{\Gamma(\nu+1)} \lim_{\alpha\to\infty} \int_0^{\alpha} \vartheta(\nu,t) e^{\nu(\ln t - t)} dt$$

$$\sim \frac{\nu^{\nu+1}}{\Gamma(\nu+1)} \cdot \frac{\Gamma(\nu+1)}{\nu^{\nu+1}} \quad \text{as} \quad \nu \to \infty \quad \text{(i.e. as } k^* \to \infty\text{)}.$$

Since $\lim_{k^*\to\infty} I_n(\eta,k^*) = 1$ and $\lim_{k^*\to\infty} I_r(\eta,k^*) = 0$, $r = 1,2,\ldots,n-1$ and

$$P(E|\eta,k^*) = \sum_{r=1}^{n} I_r(\eta,k^*),$$

$$\sup_{\substack{\eta>0 \\ k^*\geq 1}} P(E|\eta,k^*) = 1$$

and the family is outlier-prone completely on the right.

Thus for either a scale or shape change, the exchangeable model with at most one spurious observation based on the gamma distribution is outlier-prone completely on the right.

3.3 Detection of Outliers

Mount and Kale (1973) considered a general model, assuming $X_1,\ldots,X_n$ are such that $n-1$ of them are distributed with distribution function (d.f) $F(x)$ and one is distributed with d.f. $G(x)$, where F and G are stochastically ordered, i.e. $G < F$. A priori, each X_i has probability $\frac{1}{n}$ of being the spurious observation distributed as G. Let $\Psi(x) = \frac{dG}{dF}$. If $\Psi(x)$ is monotone increasing, then $u(1;n,k^*) < u(2;n,k^*) < \ldots < u(n;n,k^*)$ where $u(i;n,k^*) = P(X_{(i)}$ is the spurious observation in a sample of size n with k^* the coefficient of spuriosity).

CASE I: Scale change: Consider a situation where $n-1$ observations are from $GAM(\theta,\eta,0)$ and one is from $GAM(k^*\theta,\eta,0)$, $k^* > 0$. If $k^* = 1$ we have homogeneous data. Now

$$\Psi(x) = \frac{dG(x)}{dF(x)} = \frac{e^{-\frac{x}{\theta}\left(\frac{1-k^*}{k^*}\right)}}{k^{*\eta}}$$

and

$$\Psi'(x) = \frac{e^{-\frac{x}{\theta}\left(\frac{1-k^*}{k^*}\right)}}{k^{*\eta}} \left(\frac{k^*-1}{\theta^{k^*}}\right) \begin{cases} > 0 & \text{if } k^* > 1 \\ < 0 & \text{if } k^* < 1 \end{cases}.$$

Thus, if $k^* > 1$, $\Psi(x)$ is monotone increasing and $X_{(n)}$ has maximum probability of being spurious; if $k^* < 1$, Ψ is monotone decreasing and $X_{(1)}$ has maximum probability of being spurious.

CASE II: Shape change: If $n-1$ observations come from $GAM(\theta,\eta,0)$ and one is from $GAM(\theta,k^*\eta,0)$, $k^* > 0$, assuming the exchangeable model,

$$\Psi(x) = \frac{dG}{dF} = \frac{x^{k^*\eta-\eta}\Gamma(\eta)}{\theta^{k^*\eta-\eta}\Gamma(k^*\eta)}$$

and

$$\Psi'(x) = \frac{\Gamma(\eta)\eta(k^*-1)x^{k^*\eta-\eta-1}}{\Gamma(k^*\eta)\theta^{k^*\eta-\eta}} \begin{cases} > 0 & \text{if } k^* > 1 \\ < 0 & \text{if } k^* < 1 . \end{cases}$$

Thus, if $k^* > 1$, $\Psi(x)$ is monotone increasing and $X_{(n)}$ has maximum probability of being spurious; for $0 < k^* < 1$, $\Psi(x)$ and $u(r;n,k^*)$ are monotone decreasing and $X_{(1)}$ has maximum probabilty of being spurious.

This now shows that the spurious observation resulting from a scale or shape change is most likely to occur at the sample extremes i.e. it tends to show up as an outlier.

3.4 Estimation for Gamma parameters

3.4.1 Standard Estimators

The gamma distribution is a member of the exponential class, consequently $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} \ln X_i\right)$ are complete sufficient statistics for (θ, η).

For known shape η, the MLE $\hat{\lambda}_{MLE}$ of $\lambda = \frac{1}{\theta}$ is $\frac{\eta}{\bar{x}}$ which is biased but consistent and asymptotically $N\left(\lambda, \frac{\lambda^2}{n\eta}\right)$. Thus the UMVUE for θ (η known) is $\frac{\bar{x}}{\eta}$.

For unknown η, we obtain

$$\hat{\theta}_{MLE} = \frac{\bar{x}}{\hat{\eta}}$$

$$g(\hat{\eta}_{MLE}) = \ln \hat{\eta}_{MLE} - \psi(\hat{\eta}_{MLE}) - \ln \bar{x} + \ln \tilde{x} = 0$$

where $\tilde{x} = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$ and $\psi(z)$ is Euler's psi function i.e.

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \int_0^{\infty} \frac{\frac{e^{-t}}{t} - e^{-zt}}{1-e^{-t}}\, dt = -\gamma + \frac{1}{z} + z \sum_{i=1}^{\infty} [i(i+z)]^{-1} \quad \text{where}$$

$\gamma = .5772157$ (Euler's constant). These equations must be solved iteratively (see Choi and Wette (1969)). For large $\hat{\eta}_{MLE}$, Linhart (1965) suggested approximating $\ln \hat{\eta}_{MLE} - \psi(\hat{\eta}_{MLE})$ by $(2\hat{\eta}_{MLE} - 1/3)^{-1}$ thus

$$\hat{\eta}_{MLE} = \{(\ell n\ \bar{x} - \ell n\ \tilde{x})^{-1} + 1/3\}1/2.$$

Moment estimators give

$$\hat{\eta}_M = \frac{\left(\sum_{i=1}^{n} X_i\right)^2}{\sum_{i=1}^{n} n(X_i - \bar{X})^2}$$

$$\hat{\theta}_M = \frac{\sum_{i=1}^{n} X_i}{n\eta_M} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{\sum_{i=1}^{n} X_i} .$$

These have lower asymptotic efficiency than MLE's but Lilliefors (1971) showed that for $n \leq 20$, $\eta \geq 2$, the MSE's of the moment estimators are close to those of the MLE's. Both the MLE's and the moment estimators are biased. To approximate zero bias and smaller MSE, Lilliefors suggested the following corrections:

TABLE 3.4.1 Lilliefors' improved estimators of η and $\lambda = 1/\theta$

	Based on M.L.E.	Based on Moment Estimators
$\hat{\hat{\eta}}$	$\frac{\hat{\eta}_{MLE}}{1+3/n}$	$\frac{\hat{\eta}_M}{1+2/n} - 5/3$
$\hat{\hat{\lambda}} = \frac{\hat{\hat{1}}}{\theta}$	$\frac{n\hat{\eta}_{MLE}}{(1+\frac{3}{n})\sum_{i=1}^{n} X_i}$	$\{\frac{n\hat{\eta}_M}{(1+2/n)} - 5/3\} \frac{1}{\sum_{i=1}^{n} X_i}$

Thom (1968) has given estimators very similar to the MLE's for $\eta > 1$.

$$\eta^* = \frac{1 + \sqrt{1 + \frac{4M}{3}}}{4M}$$

$$\theta^* = \frac{\bar{x}}{\eta^*}$$

where $M = \ln(\bar{x}/\tilde{x}) = -\ln S_1$ and S_1 is sufficient for η and independent of θ. Bain and Englehardt (1975) have a chi-square approximation for M valid for all η (θ acts as a nuisance parameter).

$$2n\eta Mc \sim \chi^2_{\nu df}$$

where $W = 2n\eta M$, $c = 2\frac{E(W)}{Var(W)} = \frac{nw_1(\eta)-w_1(n\eta)}{nw_2(\eta)-w_2(n\eta)}$ and $\nu = 2\frac{[E(W)]^2}{Var(W)} = [nw_1(\eta)-w_1(n\eta)]c$ and $w_1(z) = 2z\{\ln z-\psi(z)\}$, $w_2(z) = 2z\{z\psi'(z)-1\}$ where $\psi(z)$ is the psi function. As $\eta \to 0$, $W \xrightarrow{d} \chi^2_{2(n-1)df}$.

$$\dot{w}_1(\eta) = 1 + \frac{1}{1+6\eta}$$

$$\dot{w}_2(\eta) = \begin{cases} 1 + \frac{1}{1+2.5\eta}, & 0 < \eta < 2 \\ 1 + \frac{1}{3\eta}, & \eta \geq 2 \end{cases}$$

$$\frac{\dot{\nu}}{n-1} = 1 + \frac{1}{(1+4.3\eta)^2}$$

3.4.2 Estimators suggested for use

Case I: Scale Change

Assuming the exchangeable model where n-1 observations are distributed as GAM($\theta,\eta,0$) and one is distributed as GAM($\theta k^*,\eta,0$), η known, $k^* \geq 1$, we have for $k^* = 1$, homogeneous data and the best linear unbiased estimator (BLUE) for θ is $\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} X_i}{n\eta}$ and $E(\hat{\theta}_{MLE}) = \theta$ and $MSE(\hat{\theta}_{MLE}) = Var(\hat{\theta}_{MLE}) + \{Bias(\hat{\theta}_{MLE})\}^2 = \frac{\theta^2}{n\eta}$. For the heterogeneous case ($k^*>1$), $E_{het}(\hat{\theta}_{MLE}) = \frac{1}{n\eta} \sum_{i=1}^{n} \left(\frac{n-1}{n}\theta\eta + \frac{1}{n}k^*\theta\eta\right)$ $= \frac{\theta}{n}(n-1+k^*)$ and $Bias_{het}(\hat{\theta}_{MLE}) = \frac{\theta(k^*-1)}{n}$. As $k^* \to \infty$, this bias $\to \infty$. Now $MSE_{het}(\hat{\theta}_{MLE}) = Var_{het}(\hat{\theta}_{MLE}) + \{Bias_{het}(\hat{\theta}_{MLE})\}^2$ and

$$Var_{het}(\hat{\theta}_{MLE}) = \frac{1}{(n\eta)^2} \sum_{i=1}^{n} \left\{\frac{n-1}{n}\theta^2\eta + \frac{1}{n}(k^*\theta)^2\eta\right\}$$

$$= \frac{\theta^2}{n^2\eta}\{k^{*2}+n-1\}.$$

$$\text{therefore } MSE_{het}(\hat{\theta}_{MLE}) = \frac{\theta^2}{n^2\eta}(k^{*2}+n-1) + \frac{\theta^2(k^*-1)^2}{n^2}$$

$$= \frac{\theta^2}{n^2\eta}\{k^{*2}+n-1+\eta(k^*-1)^2\}$$

As $k^* \to \infty$, $MSE_{het}(\hat{\theta}_{MLE}) \to \infty$ and hence $\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} X_i}{n\eta}$ proves to be a poor estimator of θ.

If we consider a random sample of size n where $n-m$ observations are from $GAM(\theta,\eta,0)$ and m are from $GAM(k^*\theta,\eta,0)$, $k^* > 1$ (i.e. same shape but a change in scale parameter) and a priori each subset of m observations is equally likely to be the "outlier subset", then the likelihood

$$L(\underline{x}|\theta,\eta,k^*,I) = \frac{1}{\binom{n}{m}} \frac{e^{-\sum_{x_i \notin I} \frac{x_i}{\theta}} \prod_{x_i \notin I} x_i^{\eta-1}}{\{\Gamma(\eta)\theta^{\eta}\}^{n-m}} \frac{e^{-\sum_{x_i \in I} \frac{x_i}{k^*\theta}} \prod_{x_i \in I} x_i^{\eta-1}}{\{\Gamma(\eta)(k^*\theta)^{\eta}\}^{m}}$$

$$= \frac{1}{\binom{n}{m}} \frac{\prod_{i=1}^{n} x_i^{\eta-1} e^{-\left[\sum_{x_i \notin I} \frac{x_i}{\theta} + \sum_{x_i \in I} \frac{x_i}{\theta_1}\right]}}{\{\Gamma(\eta)\}^n \theta^{\eta(n-m)} \theta_1^{\eta m}}$$

for $k^*\theta = \theta_1$ where $I = (x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ and $I \in \mathcal{I}$, the collection of all possible combinations of m observations out of n. (I represents the subset of m spurious observations). To maximize $L(\underline{x}|\theta,\eta,k^*,I)$ for $\theta,\eta > 0$, $I \in \mathcal{I}$, $k^* > 1$ we use the fact that $\Psi(x) = \frac{dG(x)}{dF(x)}$ is monotone increasing for $k^* > 1$ and hence $\max_{I \in \mathcal{I}} L(\underline{x}|\theta,\eta,k^*,I)$ occurs at $I = \hat{I} = (x_{(n-m+1)}, \ldots, x_{(n)})$. Therefore

$$L(\underline{x}|\theta,\eta,k^*,\hat{I}) = \frac{1}{\binom{n}{m}} \frac{\prod_{i=1}^{n} x_i^{\eta-1} \; e^{-\left\{\sum_{i=1}^{n-m} \frac{x_{(i)}}{\theta} + \sum_{i=n-m+1}^{n} \frac{x_{(i)}}{\theta_1}\right\}}}{\{\Gamma(\eta)\}^n \theta^{\eta(n-m)} \theta_1^{\eta m}}$$

and

$$K(\underline{x}|\theta,\eta,k^*,\hat{I}) = \ln L(\underline{x}|\theta,\eta,k^*,\hat{I})$$

$$= C + (\eta-1)\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n-m} \frac{x_{(i)}}{\theta} - \sum_{i=n-m+1}^{n} \frac{x_{(i)}}{\theta_1}$$

$$- n\ln \Gamma(\eta) - \eta(n-m)\ln \theta - \eta m \ln \theta_1 .$$

Assuming

i) θ, θ_1 and η are unknown, we obtain:

$$\frac{\partial K}{\partial \eta} = \sum_{i=1}^{n} \ln x_i - \frac{n\Gamma'(\eta)}{\Gamma(\eta)} - \{(n-m)\ln \theta + m \ln \theta_1\}$$

$$\frac{\partial K}{\partial \theta} = \sum_{i=1}^{n-m} \frac{x_{(i)}}{\theta^2} - \frac{\eta(n-m)}{\theta}$$

$$\frac{\partial K}{\partial \theta_1} = \sum_{i=n-m+1}^{n} \frac{x_{(i)}}{\theta_1^2} - \frac{\eta m}{\theta_1} .$$

Setting the above three equations equal to zero, we obtain

$$\hat{\theta} = \frac{\sum_{i=1}^{n-m} x_{(i)}}{\eta(n-m)}$$

$$\hat{\theta}_1 = \frac{\sum_{i=n-m+1}^{n} x_{(i)}}{\eta(n-m)}$$

$$h(\hat{\eta}) = \sum_{i=1}^{n} \ell n\, x_{(i)} - \frac{n\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} + n \ell n\, \hat{\eta}$$

$$- (n-m) \ell n \frac{\sum_{i=1}^{n-m} x_{(i)}}{n-m} - m \ell n \frac{\sum_{i=n-m+1}^{n} x_{(i)}}{m} = 0.$$

ii) for the case of known η, we obtain

$$\hat{\theta} = \frac{\sum_{i=1}^{n-m} x_{(i)}}{\eta(n-m)} \quad \text{and} \quad \hat{\theta}_1 = \frac{\sum_{i=n-m+1}^{n} x_{(i)}}{m\eta}$$

which are trimmed means.

iii) for the case of known θ, we may use $Y_i = \frac{X_i}{\theta}$.

$$L(\underline{y}; \eta, k^*, I) = \frac{1}{\binom{n}{m}} \frac{\prod_{i=1}^{n} y_i^{\eta-1} e^{-\{\sum_{i \notin I} y_i + \sum_{i \in I} y_i/k^*\}}}{\{\Gamma(\eta)\}^n k^{*\eta m}}$$

and

$$L(\underline{y};\eta,k^*,\hat{I}) = \frac{1}{\binom{n}{m}} \frac{\prod_{i=1}^{n} y_i^{\eta-1} e^{-\{\sum_{i=1}^{n-m} y_{(i)} + \sum_{i=n-m+1}^{n} y_{(i)}/k^*\}}}{\{\Gamma(\eta)\}^n k^{*\eta m}}$$

and

$$K(\underline{y};\eta,k^*,\hat{I}) = \ell n L(\underline{y};\eta,k^*,\hat{I})$$

$$= C' + (\eta-1)\sum_{i=1}^{n} \ell n\, y_i - \{\sum_{i=1}^{n-m} y_{(i)} + \sum_{i=n-m+1}^{n} y_{(i)}/k^*\}$$

$$- n\, \ell n\, \Gamma(\eta) - \eta m\, \ell n\, k^*.$$

Then

$$\frac{\partial K}{\partial \eta} = \sum_{i=i}^{n} \ell n\, y_i - \frac{n\Gamma'(\eta)}{\Gamma(\eta)} - m\, \ell n\, k^*$$

$$\frac{\partial K}{\partial k^*} = \sum_{i=n-m+1}^{n} \frac{y_{(i)}}{k^{*2}} - \frac{m\eta}{k^*}$$

and thus

$$\hat{k}^* = \frac{\sum_{i=n-m+1}^{n} y_{(i)}}{m\eta}$$

and

$$\frac{n\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} - m\, \ell n\, \hat{\eta} = \sum_{i=1}^{n} \ell n\, y_i - m\, \ell n \left\{ \frac{\sum_{i=n-m+1}^{n} y_{(i)}}{m} \right\} .$$

iv) for the case of known k*

$$\frac{\partial K}{\partial \eta} = \sum_{i=i}^{n} \ln x_i - \frac{n\Gamma'(\eta)}{\Gamma(\eta)} - (n-m)\ln \theta - m \ln(k^*\theta)$$

$$\frac{\partial K}{\partial \theta} = \sum_{i=i}^{n-m} \frac{x_{(i)}}{\theta^2} + \sum_{i=n-m+1}^{n} \frac{x_{(i)}}{k^*\theta^2} - \frac{\eta(n-m)}{\theta} - \frac{\eta m}{\theta} .$$

Setting $\frac{\partial K}{\partial \eta} = 0$ and $\frac{\partial K}{\partial \theta} = 0$, we obtain

$$\hat{\eta}\hat{\theta} = \frac{\sum_{i=1}^{n-m} x_{(i)} + \frac{1}{k^*} \sum_{i=n-m+1}^{n} x_{(i)}}{n}$$

$$\ln \hat{\theta} + \frac{\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} = \frac{\sum_{i=1}^{n} \ln x_i - m \ln k^*}{n} .$$

Case II. Shape change

Assuming n-1 observations distributed as GAM($\theta,\eta,0$) and one distributed as GAM($\theta,k^*\eta,0$), $k^* \geq 1$, we have, for heterogeneous data ($k^*>1$),

$$E_{het}(\hat{\theta}_{MLE}) = E(\frac{\bar{X}}{\eta}) = \frac{\theta}{n\eta} \sum_{i=1}^{n} (\frac{n-1}{n}\theta\eta + \frac{1}{n}\theta k^*\eta)$$

$$= \theta + \frac{\theta(k^*-1)}{n} .$$

As $k^* \to \infty$, bias $\to \infty$. Also

$$\text{Var}_{\text{het}}(\hat{\theta}_{\text{MLE}}) = \text{Var}\left(\frac{\bar{X}}{\eta}\right) = \frac{1}{n^2\eta^2} \sum_{i=1}^{n} \text{Var}(X_i)$$

$$= \frac{1}{n^2\eta^2} \sum_{i=1}^{n} \left(\frac{n-1}{n}\,\theta^2\eta + \frac{1}{n}\,\theta^2 k^*\eta\right)$$

$$= \frac{\theta^2}{n^2\eta}\,(n+k^*-1).$$

Then

$$\text{MSE}_{\text{het}}(\hat{\theta}_{\text{MLE}}) = \text{Var}_{\text{het}}(\hat{\theta}_{\text{MLE}}) + \{\text{Bias}\,(\hat{\theta}_{\text{MLE}})\}^2$$

$$= \frac{\theta^2(n+k^*-1)}{n^2\eta} + \frac{\theta^2\eta(k^*-1)^2}{\eta n^2}$$

$$= \frac{\theta^2\{n+k^*-1+\eta(k^*-1)^2\}}{n^2\eta}\,.$$

As $k^* \to \infty$, $\text{MSE}_{\text{het}}(\hat{\theta}_{\text{MLE}}) \to \infty$ and hence $\hat{\theta} = \frac{\bar{x}}{\eta}$ proves to be a poor estimator of θ. (We have assumed η known).

Consider now the exchangeable model where $n-m$ observations are from GAM$(\theta,\eta,0)$ and m observations are from GAM$(\theta,k^*\eta,0)$, $k^* \geq 1$ (i.e. possible change in shape; scale constant). Then

$$L(\underline{x}|\theta,\eta,k^*,I) = \frac{1}{\binom{n}{m}} \frac{e^{-\sum_{x_i \notin I} \frac{x_i}{\theta}} \prod_{x_i \notin I} x_i^{\eta-1}}{\{\Gamma(\eta)\}^{n-m}(\theta^{\eta})^{n-m}} \frac{e^{-\sum_{x_i \in I} \frac{x_i}{\theta}} \prod_{x_i \in I} x_i^{k^*\eta-1}}{\{\Gamma(k^*\eta)\}^{n-m}(\theta^{k^*\eta})^{m}}$$

$$= \frac{1}{\binom{n}{m}} \frac{e^{-T/\theta} \prod_{x_i \notin I} x_i^{\eta-1} \prod_{x_i \in I} x_i^{k^*\eta-1}}{\{\Gamma(\eta)\}^{n-m}\{\Gamma(k^*\eta)\}^{m}(\theta^{\eta})^{n-m+k^*m}}$$

where $T = \sum_{i=1}^{n} x_i$, $I = (x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ and $I \in \mathcal{J}$, the collection of all possible subsets of m observations out of n. To obtain MLE's, note that $\Psi(x) = \frac{dG}{dF}$ is monotone increasing and, by Kale (1975b), we know $\max_{I \in \mathcal{J}} L(\underline{x}|\theta,\eta,k^*,I)$ occurs at $\hat{I} = I(\theta,\eta,k^*$ fixed) where $\hat{I} = (x_{(n-m+1)}, \ldots, x_{(n)})$. Thus

$$\max_{\substack{k^*>1 \\ I \in \mathcal{J}}} L(\underline{x}|\theta,\eta,k^*,I) = \max_{k^* \geq 1} L(\underline{x}|\theta,\eta,k^*,\hat{I})$$

and

$$L(\underline{x}|\theta,\eta,k^*,\hat{I}) = \frac{e^{-T/\theta} \prod_{i=1}^{n-m} x_{(i)}^{\eta-1} \prod_{i=n-m+1}^{n} x_{(i)}^{k^*\eta-1}}{\binom{n}{m}\{\Gamma(\eta)\}^{n-m}\{\Gamma(k^*\eta)\}^{m}(\theta^{\eta})^{n-m+k^*m}} .$$

Using $\eta_1 = k^*\eta$ ($> \eta$ since $k^* > 1$)

$$K(\underline{x}|\theta,\eta,\eta_1,\hat{I}) = \ell n\, L(\underline{x}|\theta,\eta,\eta_1,\hat{I})$$

$$= C - \frac{T}{\theta} + (\eta-1) \sum_{i=1}^{n-m} \ell n\, x_{(i)} + (\eta_1-1) \sum_{i=n-m+1}^{n} \ell n\, x_{(i)}$$

$$- (n-m)\ln \Gamma(\eta) - m \ln \Gamma(\eta_1) - (n-m)\eta \ln \theta - m\eta_1 \ln \theta$$

and

i) assuming θ, η, and η_1 unknown, we obtain

$$\frac{\partial K}{\partial \theta} = \frac{T}{\theta^2} - \frac{(n-m)\eta}{\theta} - \frac{m\eta_1}{\theta}$$

$$\frac{\partial K}{\partial \eta} = \sum_{i=1}^{n-m} \ln x_{(i)} - \frac{(n-m)\Gamma'(\eta)}{\Gamma(\eta)} - (n-m) \ln \theta$$

$$\frac{\partial K}{\partial \eta_1} = \sum_{i=n-m+1}^{n} \ln x_{(i)} - \frac{m\Gamma'(\eta_1)}{\Gamma(\eta_1)} - m \ln \theta \ .$$

Now $\frac{\partial K}{\partial \theta} = 0$ implies $\hat{\theta} = \frac{T}{(n-m)\hat{\eta} + m\hat{\eta}_1}$ and

$$\frac{\partial K}{\partial \eta} = 0 \quad \text{implies} \quad \frac{\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} + \ln \hat{\theta} = \frac{\sum_{i=1}^{n-m} \ln x_{(i)}}{n-m} \quad \text{and}$$

$$\frac{\partial K}{\partial \eta_1} = 0 \text{ implies } \frac{\Gamma'(\hat{\eta}_1)}{\Gamma(\hat{\eta}_1)} + \ln \hat{\theta} = \frac{\sum_{i=n-m+1}^{n} \ln x_{(i)}}{m} \ .$$

ii) for the case of known shape parameter η,

$$\frac{\partial K}{\partial \theta} = \frac{T}{\theta^2} - \frac{(n-m)\eta}{\theta} - \frac{mk^*\eta}{\theta} \quad \text{and}$$

$$\frac{\partial K}{\partial k^*} = \eta \sum_{i=n-m+1}^{n} \ln x_{(i)} - m\eta \frac{\Gamma'(k^*\eta)}{\Gamma(k^*\eta)} - m\eta \ln \theta$$

and setting these equal to zero, we obtain

$$\hat{\theta} = \frac{T}{\eta(n-m+mk^*)} \qquad \text{and}$$

$$\ell(\hat{k}^*) = \frac{\Gamma'(\hat{k}^*\eta)}{\Gamma(\hat{k}^*\eta)} + \ln \hat{\theta} - \frac{\sum_{i=n-m+1}^{n} \ln x_{(i)}}{m} = 0 .$$

iii) for the case of known θ, we may use $Y_i = \frac{X_i}{\theta}$ and hence obtain

$$\frac{\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} = \frac{\sum_{i=1}^{n-m} \ln y_{(i)}}{n-m}$$

and

$$\frac{\Gamma'(\hat{\eta}_1)}{\Gamma(\hat{\eta}_1)} = \frac{\sum_{i=n-m+1}^{n} \ln y_{(i)}}{m} .$$

iv) for the case of known k^*

$$\frac{\partial K}{\partial \theta} = \frac{T}{\theta^2} - \frac{(n-m)\eta}{\theta} - \frac{mk^*\eta}{\theta}$$

$$\frac{\partial K}{\partial \eta} = \sum_{i=1}^{n-m} \ln x_{(i)} + k^* \sum_{i=n-m+1}^{n} \ln x_{(i)} - \frac{(n-m)\Gamma'(\eta)}{\Gamma(\eta)} - \frac{mk^*\Gamma'(k^*\eta)}{\Gamma(k^*\eta)}$$

and thus $\hat{\theta} = \frac{T}{\hat{\eta}\{n-m(1-k^*)\}}$ and

$$\sum_{i=1}^{n-m} \ln x_{(i)} + k^* \sum_{i=n-m+1}^{n} \ln x_{(i)} - \frac{(n-m)\Gamma'(\hat{\eta})}{\Gamma(\hat{\eta})} - \frac{mk^*\Gamma'(k^*\hat{\eta})}{\Gamma(k^*\hat{\eta})} = 0.$$

CHAPTER IV

The Lognormal Distribution

We now examine the lognormal distribution as a life-testing distribution that is a competitor to the gamma distribution. We shall show that the exchangeable model involving the lognormal family of distributions indexed by the shape parameter σ is outlier-resistant completely on the right. We shall then determine that as heterogeneity increases the spurious observation tends to appear as an outlier and we shall determine where it will most likely appear. We shall also consider estimation in the presence of this outlier.

4.1 Characteristics of the Lognormal Distribution.

If a random variable X has the lognormal distribution with location parameter μ and shape parameter σ, then the probability density function of X is given by

$$f(x;\mu,\sigma) = \begin{cases} \frac{1}{\sigma x\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma^2}(\ln x-\mu)^2\}, & x > 0,\ \sigma > 0,\ -\infty < \mu < \infty \\ 0, & \text{elsewhere} \end{cases}$$

We shall write $X \sim \Lambda(\mu,\sigma)$. The following table summarizes some some of its properties:

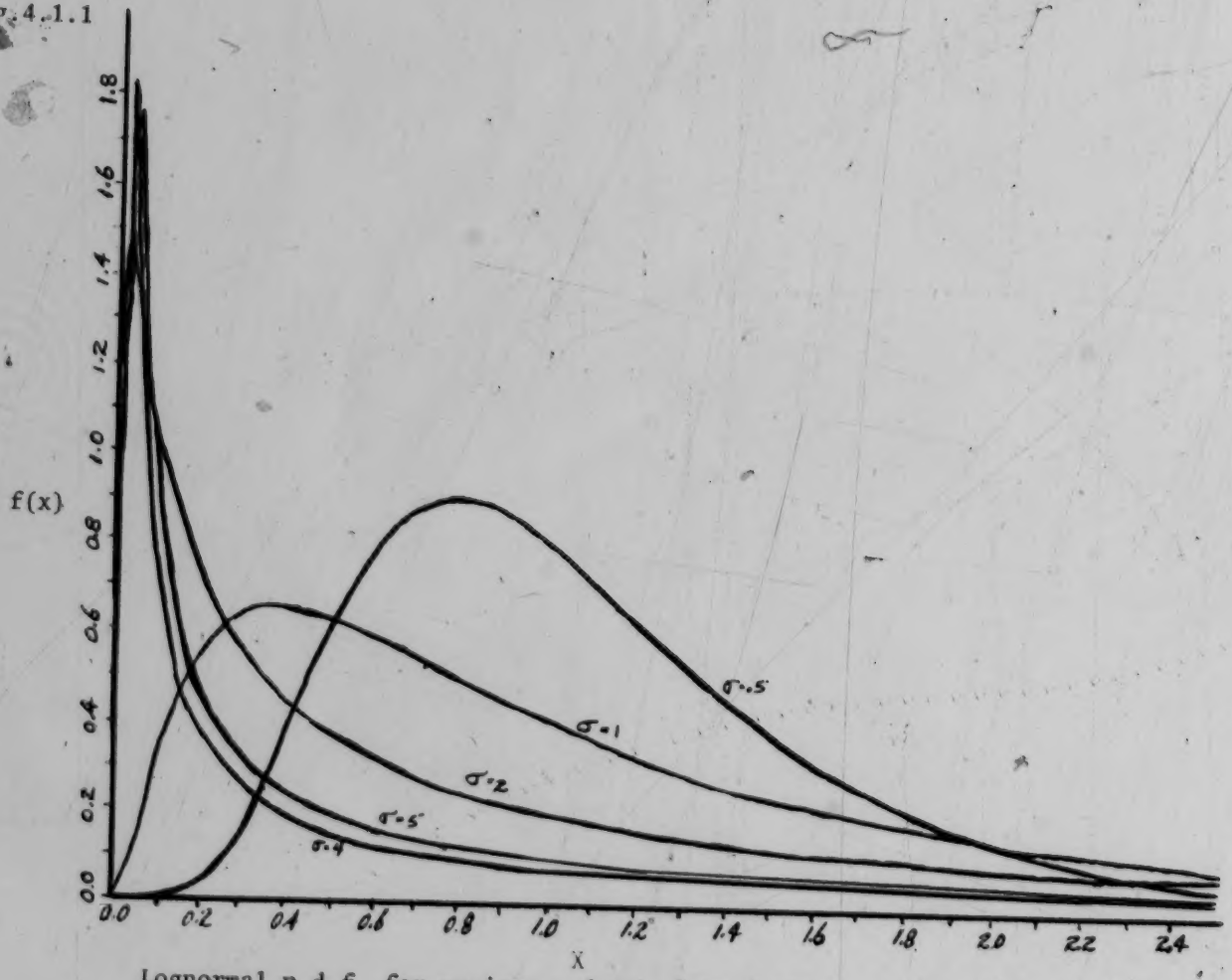
Table 4.1.1

mean	$e^{\mu+\frac{1}{2}\sigma^2}$
variance	$e^{\sigma^2}(e^{\sigma^2}-1)e^{2\mu}$
mode	$e^{\mu-\sigma^2}$
median	e^{μ}
coefficient of variation	$(e^{\sigma^2}-1)^{1/2}$
skewness	$(e^{\sigma^2}+2)(e^{\sigma^2}-1)^{1/2}$
kurtosis	$\omega^4+2\omega^3+3\omega^2-6$, $\omega = e^{\sigma^2}$

The lognormal distribution has monotone likelihood ratio (MLR) in $T_1(x) = \ln x$, σ known and in $T_2(x) = (\ln x-\mu)^2$, μ known but not

MLR in x for μ known and nonzero. $T_1 = \sum_{i=1}^{n} \ln x_i$ is a complete sufficient statistic for μ (σ known) and ($T_2 = \sum_{i=1}^{n} (\ln x_i)^2$, $T_1 = \sum_{i=1}^{n} \ln x_i$) are jointly complete sufficient statistics for (μ, σ^2). $T_3 = \sum_{i=1}^{n} (\ln x_i - \mu)^2$ is sufficient for σ^2(μ known). The lognormal distribution can assume shapes from severely right-skewed to essentially symmetrical.

Fig.4.1.1



Lognormal p.d.f. for various values of σ (μ =0)

If $X \sim \Lambda(\mu, \sigma)$ then $Y = \ln X \sim N(\mu, \sigma^2)$.

For some data, the lognormal distribution is a competitor to the gamma distribution. The lognormal p.d.f. is unimodal, vanishes at $x = 0$ and its mode is at $x = e^{-\sigma^2} < 1$. As σ^2 increases, the mode converges to zero. From the above graph where $\sigma = 2$, the lognormal p.d.f. appears monotonically decreasing for almost all $x > 0$ and when $\sigma = 5$ the mode is virtually zero. This may be compared to the gamma distribution with shape parameter $\eta < 1$. Here the density is infinite at zero and monotonically decreasing thereafter. The lognormal distribution has a non-monotonic failure rate.

Neyman and Scott (1971) document data from rain-making experiments where the data is nonzero rainfall per experimental unit (an experimental day or storm). Here the distributions are reverse J-shaped with long "tails" and frequently display substantial "outliers". They suggest that an outlier-prone distribution such as the gamma or lognormal might appropriately model this data.

Nelson (1977) points out the usefulness of the lognormal distribution for approximating distributions of input variables such as costs, sales, market shares, etc. required for Monte Carlo simulations of business decisions. This distribution has been used by Howard and Dodson (1961) and Peck (1961) to study semiconductor devices and by Goldwaithe (1961) in small-particle statistical economics and biology. Singpurwalla and Keubler (1966) used it to study the lifetimes of high speed steel drills since the failure rate first increases and then decreases, indicating the drill could resharpen itself and prolong its life. It has wide applicability to reliability, especially maintainablity and fracture problems. If $X_1 < X_2 < \ldots < X_n$ denote

sizes of a fatigue crack at successive stages of its growth and X_0 the initial size of the crack, assuming a "proportional effect model" for growth of the crack (Kao (1965)), this implies crack growth at stage i, $X_i - X_{i-1}$, is randomly proportional to the size of the crack, X_{i-1} (i.e. $X_i - X_{i-1} = \pi_i X_{i-1}$, $i = 1,2,\ldots,n$ where π_i are independent), and the item fails when crack size reaches X_n. It can be shown that $\ln X_n$ is asymptotically normal and hence X_n is lognormal. This model is also true for the distribution of oil pool sizes by the same argument on how the pools are initially formed.

4.2 Outlier-proneness of the exchangeable model with the Lognormal distribution

Neyman and Scott (1971) have demonstrated that for i.i.d.r.v.'s the family of lognormal distributions indexed by σ is outlier-prone completely on the right. If we now consider the exchangeable model with at most one outlier, we have n-1 observations from $\Lambda(\mu,\sigma)$ and one observation from $\Lambda(\mu_1,\sigma)$, $\mu_1 \geq \mu$ (Case I) or n-1 observations from $\Lambda(\mu,\sigma)$ and one observation from $\Lambda(\mu,\sigma_1)$, $\sigma_1 \geq 0$ (Case II).

Case I: Scale change

We first consider the exchangeable model based on the lognormal distribution with possible change in μ. Then $f(x;\underline{\theta})$ is $\Lambda(\mu,\sigma)$ and $f(x;\underline{\xi})$ is $\Lambda(\mu_1,\sigma)$ where $\mu_1 = k^*\mu$, $k^* \geq 1$. Kale (1975b) proved that the exchangeable model with (at most) one possible outlier observation for scale parameter families for non-negative random variables involving a possible change in scale is outlier-prone completely on the right. Thus the model we are considering would be outlier-prone.

Case II: Shape Change

Consider the exchangeable model based on the lognormal distribution with possible change in shape parameter σ. Then $f(x;\underline{\theta})$ is $\Lambda(0,\sigma)$ and $f(x;\underline{\xi})$ is $\Lambda(0,\sigma_1)$ where $\sigma_1^2 = k^*\sigma^2$, $k^* \geq 1$. The likelihood may be written as

$$L(\underline{x};\sigma,k^*) = \frac{1}{n}\sum_{r=1}^{n}\prod_{i\neq r} f(x_i;\underline{\theta})f(x_r;\underline{\xi}),\ x_i > 0\ ,\ k^* \geq 1,$$

$$\sigma_1^2 = k^*\sigma^2\ ,\ i = 1,2,\ldots,n$$

($k^* \geq 1$ since we are considering $x_{(n)}$ as a possible outlier). Then the joint density of the order statistics may be written as

$$f(x_{(1)},\cdots,x_{(n)}) = \frac{1}{n}\, n!\ \sum_{r=1}^{n}\frac{f(x_r;\sigma_1)}{f(x_r;\sigma)}\prod_{i=1}^{n} f(x_i,\sigma)$$

and

$$g(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= \int_{x_{(1)}}^{x_{(n-1)}}\int_{x_{(1)}}^{x_{(n-2)}}\cdots\int_{x_{(1)}}^{x_{(3)}} f(x_{(1)},\ldots,x_{(n)})dx_{(2)}\cdots dx_{(n-2)}$$

$$= (n-1)!\ \sum_{r=1}^{n} h_r(x_{(1)},x_{(n-1)},x_{(n)})$$

where $h_r(x_{(1)},x_{(n-1)},x_{(n)}) = \int_{S_{n-3}}\frac{f(x_r;\sigma_1)}{f(x_r;\sigma)}\prod_{i=1}^{n} f(x_i;\sigma)dx_2\ \cdots dx_{n-2}$

and S_{n-3} is the region $x_{(1)} < x_{(2)} < \cdots < x_{(n-2)} < x_{(n-1)}$.

For $r = 2,\ldots,n-2$

$$h_r(x_{(1)},x_{(n-1)},x_{(n)}) = \frac{f(x_{(1)};\sigma)f(x_{(n-1)};\sigma)f(x_{(n)};\sigma)}{(r-2)!(n-r-2)!}$$

(continued)

$$\cdot \int_{x_{(1)}}^{x_{(n-1)}} [F(x_{(r)};\sigma)-F(x_{(1)};\sigma)]^{r-2}[F(x_{(n-1)};\sigma)-F(x_{(r)};\sigma)]^{n-r-2}$$

$$f(x_{(r)};\sigma_1)dx_{(r)} ,$$

while

$$h_1(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= f(x_{(1)};\sigma_1)f(x_{(n-1)};\sigma)f(x_{(n)};\sigma) \frac{[F(x_{(n-1)};\sigma)-F(x_{(1)};\sigma)]^{n-3}}{(n-3)!} ,$$

$$h_{n-1}(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= f(x_{(1)};\sigma)f(x_{(n-1)};\sigma_1)\ f(x_{(n)};\sigma) \frac{[F(x_{(n-1)};\sigma)-F(x_{(1)};\sigma)]^{n-3}}{(n-3)!} ,$$

and

$$h_n(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= f(x_{(1)};\sigma)f(x_{(n-1)};\sigma)f(x_{(n)};\sigma_1) \frac{[F(x_{(n-1)};\sigma)-F(x_{(1)};\sigma)]^{n-3}}{(n-3)!} .$$

Thus, letting $t(y) = f(y;\sigma)$ and $s(y) = f(y;\sigma_1)$

$$(4.2.1) \quad \ldots . \quad g(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= (n-1)!\{t(x_{(1)})t(x_{(n-1)})t(x_{(n)})$$

$$\times \sum_{r=2}^{n-2} \int_{x_{(1)}}^{x_{(n-1)}} \frac{[T(x_{(r)})-T(x_{(1)})]^{r-2}[T(x_{(n-1)})-T(x_{(r)})]^{n-r-2}s(x_{(r)})}{(r-2)!(n-r-2)!}\, dx_{(r)}$$

$$+[s(x_{(1)})t(x_{(n-1)})t(x_{(n)})+t(x_{(1)})s(x_{(n-1)})t(x_{(n)})$$

$$+t(x_{(1)})t(x_{(n-1)})s(x_{(n)})] \frac{[T(x_{(n-1)})-T(x_{(1)})]^{n-3}}{(n-3)!}\} .$$

In 4.2.1, we may replace the letter of integration in the definite integrals by z and use the fact that

$$(n-1)! \sum_{r=2}^{n-2} \frac{\alpha^{r-2}\beta^{n-r-2}}{(r-2)!(n-r-2)!} = \frac{(n-1)!}{(n-4)!}(\alpha+\beta)^{n-4}$$

to give

$$g(x_{(1)},x_{(n-1)},x_{(n)}) = \frac{(n-1)!}{(n-3)!}\{(n-3)t(x_{(1)})t(x_{(n-1)})t(x_{(n)}) \times$$

$$[T(x_{(n-1)})-T(x_{(n)})]^{n-4}[S(x_{(n-1)})-S(x_{(1)})]$$

$$+ [s(x_{(1)})t(x_{(n-1)})t(x_{(n)})+t(x_{(1)})s(x_{(n-1)})t(x_{(n)})$$ (continued)

$$+ t(x_{(1)})t(x_{(n-1)})s(x_{(n)})][T(x_{(n-1)})-T(x_{(1)})]^{n-3}\}$$

$$= \frac{(n-1)!}{(n-3)!}\{T(x_{(n-1)})-T(x_{(1)})\}^{n-4}\{(n-3)t(x_{(1)})t(x_{(n-1)})t(x_{(n)})\times$$

$$[S(x_{(n-1)})-S(x_{(1)})] + [s(x_{(1)})t(x_{(n-1)})t(x_{(n)})+t(x_{(1)})s(x_{(n-1)})t(x_{(n)})$$

$$+ t(x_{(1)})t(x_{(n-1)})s(x_{(n)})][T(x_{(n-1)})-T(x_{(1)})]\}.$$

Let

$$p = P\{X_{(n)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

$$= \int_0^\infty \int_0^{x_{(n-1)}} \int_{(k+1)x_{(n-1)}-kx_{(1)}}^\infty g(x_{(1)},x_{(n-1)},x_{(n)})dx_{(n)}dx_{(1)}dx_{(n-1)}$$

and let $x = x_{(1)}$ and $y = x_{(n-1)}$.

Theorem 4.2.1: If $p = P\{X_{(n)} > (k+1)X_{(n-1)} - kX_{(1)}\}$ then, for non-negative random variables,

$$p \leq 1-k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

where S and s are, respectively, the distribution function and probability density function of the spurious observation and T and t

are, respectively, the distribution function and probability density function of the non-spurious observations.

Proof:

$$p = P\{X_{(n)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

$$= \int_0^{\infty}\int_0^{x_{(n-1)}}\int_{(k+1)x_{(n-1)}-kx_{(1)}}^{\infty} g(x_{(1)},x_{(n-1)},x_{(n)})dx_{(n)}dx_{(1)}dx_{(n-1)}$$

$$= \frac{(n-1)!}{(n-3)!}\int_0^{\infty}\int_0^{x_{(n-1)}} \{T(x_{(n-1)})-T(x_{(1)})\}^{n-4}$$

$$\{(n-3)t(x_{(1)})t(x_{(n-1)})[1-T\{(k+1)x_{(n-1)}-kx_{(1)}\}][S(x_{(n-1)})-S(x_{(1)})]$$

$$+ \{s(x_{(1)})t(x_{(n-1)})[1-T\{(k+1)x_{(n-1)}-kx_{(1)}\}]$$

$$+ t(x_{(1)})s(x_{(n-1)})[1-T\{(k+1)x_{(n-1)}-kx_{(1)}\}]$$

$$+ t(x_{(1)})t(x_{(n-1)})[1-S\{(k+1)x_{(n-1)}-kx_{(1)}\}]\}$$

$$\times [T(x_{(n-1)})-T(x_{(1)})]\}dx_{(1)}dx_{(n-1)}$$

$$= \frac{(n-1)!}{(n-3)!}\{I_1+I_2+I_3+I_4 - (I_5+I_6+I_7+I_8)\}.$$

Let $x = x_{(1)}$, $y = x_{(n-1)}$

$$I_1 = \int_0^\infty \int_0^y (n-3)t(x)t(y)\{T(y)-T(x)\}^{n-4}\{S(y)-S(x)\}dxdy$$

$$= (n-3)\int_0^\infty S(y)t(y)\int_0^y t(x)\{T(y)-T(x)\}^{n-4}dxdy$$

$$-(n-3)\int_0^\infty t(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-4}dydx$$

$$= (n-3)\int_0^\infty S(y)t(y)\frac{\{T(y)\}^{n-3}}{(n-3)}\,dy - (n-3)\int_0^\infty S(x)t(x)\frac{\{1-T(x)\}^{n-3}}{(n-3)}\,dx$$

$$= \int_0^\infty t(x)S(x)[\{T(x)\}^{n-3}-\{1-T(x)\}^{n-3}]dx \quad .$$

$$I_2 = \int_0^\infty \int_0^y \{T(y)-T(x)\}^{n-3}s(x)t(y)dxdy$$

$$= \int_0^\infty s(x)\int_x^\infty \{T(y)-T(x)\}^{n-3}t(y)dydx$$

$$= \int_0^\infty \frac{s(x)}{(n-2)}\{1-T(x)\}^{n-2}dx.$$

Integrating by parts where $u = \{1-T(x)\}^{n-2}$ $\quad dv = s(x)dx$

$$du = -(n-2)t(x)\{1-T(x)\}^{n-3}dx \quad v = S(x)$$

then

$$I_2 = \int_0^\infty S(x)t(x)\{1-T(x)\}^{n-3}dx.$$

$$I_3 = \int_0^\infty \int_0^y \{T(y)-T(x)\}^{n-3}t(x)s(y)dxdy$$

$$= \int_0^\infty s(y) \int_0^y \{T(y)-T(x)\}^{n-3}t(x)dxdy$$

$$= \int_0^\infty s(x) \frac{\{T(x)\}^{n-2}}{(n-2)} dx \quad .$$

Integrating by parts where $u = \{T(x)\}^{n-2}$ $\quad dv = s(x)dx$

$$du = (n-2)\{T(x)\}^{n-3}t(x)dx \quad v = S(x)$$

$$I_3 = \frac{1}{n-2} - \int_0^\infty S(x)t(x)\{T(x)\}^{n-3}dx \quad .$$

$$I_4 = \int_0^\infty \int_0^y t(x)t(y)\{T(y)-T(x)\}^{n-3}dx$$

$$= \int_0^\infty t(x) \int_x^\infty t(y)\{T(y)-T(x)\}^{n-3}dydx$$

$$= \int_0^\infty t(x) \frac{\{1-T(x)\}^{n-2}}{(n-2)} dx$$

$$= \frac{1}{(n-1)(n-2)} \cdot$$

Therefore $I_1 + I_2 + I_3 + I_4 = \frac{1}{(n-2)} + \frac{1}{(n-1)(n-2)}$

$$= \frac{n}{(n-1)(n-2)} \cdot$$

$$I_5 = \int_0^\infty \int_0^y (n-3)t(x)t(y)\{T(y)-T(x)\}^{n-4}\, T\{(k+1)y-k(x)\}\{S(y)-S(x)\}dxdy$$

$$\geq \int_0^\infty \int_0^y (n-3)t(x)t(y)\{T(y)-T(x)\}^{n-4} T(y)\{S(y)-S(x)\}dxdy$$

$$= \int_0^\infty (n-3)t(y)T(y)S(y)\int_0^y t(x)\{T(y)-T(x)\}^{n-4}dxdy$$

$$- \int_0^\infty (n-3)t(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-3}dydx$$

$$- \int_0^\infty (n-3)t(x)T(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-4}dydx$$

$$= \int_0^\infty t(x)T(x)S(x)\{T(x)\}^{n-3}dx - \int_0^\infty \frac{(n-3)}{(n-2)}\, t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$- \int_0^\infty t(x)T(x)S(x)\{1-T(x)\}^{n-3}dx.$$

Therefore

$$I_5 \geq \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx - \frac{(n-3)}{(n-2)} \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$- \int_0^\infty t(x)T(x)S(x)\{1-T(x)\}^{n-3}dx \quad .$$

$$I_7 \geq \int_0^\infty \int_0^y t(x)s(y)T(y)\{T(y)-T(x)\}^{n-3}dxdy$$

$$= \int_0^\infty s(y)T(y)\int_0^y t(x)\{T(y)-T(x)\}^{n-3}dxdy$$

$$= \int_0^\infty s(x)T(x)\frac{\{T(x)\}^{n-2}}{(n-2)}dx$$

$$= \int_0^\infty s(x)\frac{\{T(x)\}^{n-1}}{(n-2)}dx \quad .$$

Integrating by parts where $u = \{T(x)\}^{n-1}$ $\quad dv = s(x)dx$

$$du = (n-1)t(x)\{T(x)\}^{n-2}dx \quad v = S(x)$$

$$I_7 \geq \frac{1}{n-2} - \frac{(n-1)}{(n-2)} \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx \quad .$$

$$I_8 \geq \int_0^\infty \int_0^y t(x)t(y)S(y)\{T(y)-T(x)\}^{n-3}dxdy$$

$$= \int_0^\infty \frac{t(y)S(y)}{(n-2)}\{T(y)\}^{n-2}dy$$

$$= \frac{1}{(n-2)} \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx.$$

$$I_6 = \int_0^\infty t(y)\int_0^y s(x)\{T(y)-T(x)\}^{n-3}T\{(k+1)y-kx\}dxdy \quad .$$

Integrating by parts where

$$u = \{T(y)-T(x)\}^{n-3}T\{(k+1)y-kx\} \qquad dv = s(x)dx$$

$$du = \{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}\{-kdx\} \qquad v = S(x)$$

$$+ \ (n-3)\{T(y)-T(x)\}^{n-4}T\{(k+1)y-kx\}\{-t(x)\}dx$$

$$I_6 = k \int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$+ (n-3)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-4}T\{(k+1)y-kx\}t(x)dxdy$$

$$\geq k \int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$+ (n-3)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-4}T(y)t(x)dxdy \ .$$

But

$$(n-3)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-4}T(y)t(x)dxdy$$

$$= (n-3)\int_0^\infty t(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-3}dydx$$

$$+ (n-3)\int_0^\infty t(x)T(x)S(x)\int_x^\infty t(y)\{T(y)-T(x)\}^{n-4}dydx$$

$$= \frac{(n-3)}{(n-2)}\int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$+ \int_0^\infty t(x)T(x)S(x)\{1-T(x)\}^{n-3}dx$$

$$= \frac{(n-3)}{(n-2)}\int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx - \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$+ \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-3}dx.$$

Therefore $I_5 + I_6 + I_7 + I_8 \geq \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx$

$$- \frac{(n-3)}{(n-2)}\int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx + \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

(continued)

$$- \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-3}dx$$

$$+ k\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$+ \frac{(n-3)}{(n-2)} \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx - \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-2}dx$$

$$+ \int_0^\infty t(x)S(x)\{1-T(x)\}^{n-3}dx + \frac{1}{(n-2)}$$

$$- \frac{(n-1)}{(n-2)} \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx \quad + \frac{1}{(n-2)} \int_0^\infty t(x)S(x)\{T(x)\}^{n-2}dx$$

Therefore

$$p \leq (n-1)(n-2)\left[\frac{n}{(n-1)(n-2)} - \frac{1}{(n-2)}\right.$$

$$\left. - k\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy\right]$$

$$= 1 - k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)[T(y)-T(x)]^{n-3}t\{(k+1)y-kx\}dxdy \ .$$

Theorem 4.2.2: The exchangeable model with (at most) one spurious observation based on the lognormal family of distributions indexed by shape parameter σ is outlier-resistant completely on the right.

Proof: From Theorem 4.2.1 we know that if

$$p = P\{X_{(n)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

then, for non-negative random variables,

$$p \leq 1 - k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

where $x = x_{(1)}$, $y = x_{(n-1)}$, S and s are, respectively, the distribution function and p.d.f. of a spurious observation, and T and t are, respectively, the distribution function and p.d.f. of a non-spurious observation. If, for all $k > 0$, $n > 2$ $\sup p < 1$ then the family is outlier-resistant completely on the right.

It is then sufficient to show that

$$(4.2.2)\quad H = k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$> 0.$$

Let $s_1 = \frac{\ln x}{\sigma}$ $\qquad$ $s_2 = \frac{\ln y}{\sigma}$ $\qquad$ $v = \frac{\sigma}{\sigma_1}$

$$ds_1 = \frac{dx}{x\sigma} \qquad ds_2 = \frac{dy}{\sigma y}$$

If ϕ and Φ respectively denote the standardized normal p.d.f. and d.f. then expression (4.2.2) becomes

$$k(n-1)(n-2)\int_{-\infty}^{\infty}\phi(s_2)\int_{-\infty}^{s_2}\Phi(vs_1)\{\Phi(s_2)-\Phi(s_1)\}^{n-3}\phi\{\frac{\ell n[(k+1)e^{\sigma s_2}-ke^{\sigma s_1}]}{\sigma}\}$$

$$\times\frac{e^{\sigma s_1}ds_1ds_2}{(k+1)e^{\sigma s_2}-ke^{\sigma s_1}}$$

$$\geq k(n-1)(n-2)\int_{3}^{5}\phi(s_2)\int_{0}^{2}\frac{\Phi(vs_1)e^{\sigma s_1}}{(k+1)e^{\sigma s_2}-ke^{\sigma s_1}}\{\Phi(s_2)-\Phi(s_1)\}^{n-3}$$

$$\times\phi\{\frac{\ell n[(k+1)e^{\sigma s_2}-ke^{\sigma s_1}]}{\sigma}\}ds_1ds_2$$

In this region, $s_2 \geq 3$, $s_1 \geq 0$, $s_2 \geq 1 + s_1$. Therefore $\Phi(vs_1) \geq 1/2$ for all v (i.e. for all σ_1), $e^{\sigma s_1} \geq 1$,

$$\Phi(s_2) - \Phi(s_1) \geq \Phi(3) - \Phi(2) \text{ and } e^{-\sigma s_2} \geq e^{-5\sigma}.$$

Then

$$H \geq \frac{k(n-1)(n-2)}{2(k+1)}\{\Phi(3) - \Phi(2)\}^{n-3}\frac{1}{2}e^{-5\sigma}\,2\int_{3}^{5}\phi(s_2)\phi(s_2+\frac{\ell n(k+1)}{\sigma})ds_2 .$$

Let $u = \frac{\ell n(k+1)}{\sigma}$. Then $u > 0$ since $k > 0$ and

$$\int_{3}^{5}\phi(s_2)\phi(s_2+u)ds_2 = \frac{1}{\sqrt{2\pi}}\int_{3}^{5}\frac{1}{\sqrt{2\pi}}e^{-1/2[\frac{s_2+u/2}{1/\sqrt{2}}]^2-u^2/4}ds_2$$

$$= \frac{e^{-u^2/4}}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \int_{\sqrt{2}(3+u/2)}^{\sqrt{2}(5+u/2)} \phi(z)dz$$

$$= \frac{e^{-u^2/4}}{2\sqrt{\pi}} \left[\Phi\{\sqrt{2}(5+u/2)\} - \Phi\{\sqrt{2}(3+u/2)\}\right]$$

$$> 0.$$

Then for all $k > 0$, $n > 2$, $v = \sigma/\sigma_1 > 0$, $H > 0$ and $p = 1-H < 1$ and therefore $\sup p < 1$. Thus the exchangeable model with (at most) one spurious observation involving the lognormal family of distributions indexed by shape parameter σ is outlier-resistant completely on the right.

4.3 Detection of Outliers

Case I: Scale change.

If $n-1$ observations are from $\Lambda(\mu,\sigma)$ and one observation is from $\Lambda(\mu_1,\sigma)$, $\mu_1 = k^*\mu$, $k^* \geq 1$ then

$$\Psi(x) = \frac{dG(x)}{dF(x)} = \frac{\frac{1}{\sigma x\sqrt{2\pi}}\, e^{-1/2\left(\frac{\ln x-\mu_1}{\sigma}\right)^2}}{\frac{1}{\sigma x\sqrt{2\pi}}\, e^{-1/2\left(\frac{\ln x-\mu}{\sigma}\right)^2}}$$

$$= \exp\left[-\frac{1}{2\sigma^2}\{(\ln x-\mu_1)^2-(\ln x-\mu)^2\}\right]$$

$$= \left[\exp\left\{-\frac{(\mu_1^2-\mu^2)}{2\sigma^2}\right\}\right] x^{\frac{\mu_1-\mu}{\sigma^2}}$$

and $\Psi'(x) = \left(\frac{\mu_1-\mu}{\sigma^2}\right) x^{\frac{\mu_1-\mu}{\sigma^2}-1} \exp\left\{-\frac{1}{2\sigma^2}(\mu_1^2-\mu^2)\right\}$.

Thus $\Psi'(x)$ $\begin{cases} > 0 & \text{if } \mu_1 > \mu \\ < 0 & \text{if } \mu_1 < \mu \end{cases}$.

Consequently $X_{(n)}$ has maximum probability of being the spurious observation if $\mu_1 > \mu$; $X_{(1)}$ has maximum probability of being the

spurious observation if $\mu_1 < \mu$.

Case II: Shape change

If $n-1$ observations are from $\Lambda(\mu, \sigma)$ and one observation is from $\Lambda(\mu, \sigma_1)$, $\sigma_1^2 = k^*\sigma^2$, $k^* \geq 1$ then

$$\Psi(x) = \frac{dG(x)}{dF(x)} = \frac{\sigma}{\sigma_1} \exp\left\{-\frac{1}{2}(\ln x-\mu)^2\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma^2}\right)\right\}$$

is not monotone in x.

Consider $u(r;n,\sigma,k^*) = P(X_{(r)}$ is the spurious observation) (i.e. $X_{(r)}$ is the order statistic whose distribution has parameter σ_1). Then

$$u(r;n,\sigma,k^*) = \binom{n-1}{r-1}\int_0^\infty \{F(y;\sigma)\}^{r-1}\{1-F(y;\sigma)\}^{n-r} f(y;\sigma_1)dy$$

$$= \binom{n-1}{r-1}\int_0^\infty \left\{\Phi\left(\frac{\ln y-\mu}{\sigma}\right)\right\}^{r-1}\left\{1-\Phi\left(\frac{\ln y-\mu}{\sigma}\right)\right\}^{n-r} \frac{\exp -\frac{1}{2}\left(\frac{\ln y-\mu}{\sigma_1}\right)^2}{\sigma_1 y\sqrt{2\pi}}dy$$

where Φ is the standard normal distribution function

$$= \binom{n-1}{r-1} \int_0^{\infty} \left[\Phi\left\{\left(\frac{\ln y-\mu}{\sigma_1}\right)\sqrt{k^*}\right\}\right]^{r-1} \left[1-\Phi\left\{\left(\frac{\ln y-\mu}{\sigma}\right)\sqrt{k^*}\right\}\right]^{n-r}$$

$$\frac{\exp - \frac{1}{2}\left(\frac{\ln y-\mu}{\sigma_1}\right)^2}{\sigma_1 y\sqrt{2\pi}} dy$$

and setting $x = \frac{\ln y-\mu}{\sigma_1}$ we have

$$u(r;n,\sigma,k^*) = \binom{n-1}{r-1} \int_{-\infty}^{\infty} \{\Phi(x\sqrt{k^*})\}^{r-1} \{1-\Phi(x\sqrt{k^*})\}^{n-r} \frac{\exp\{-\frac{x^2}{2}\}}{\sqrt{2\pi}} dx .$$

Now

$$0 \leq \Phi(x\sqrt{k^*}) \leq 1 \quad \text{and} \quad \lim_{k^*\to\infty} \Phi(x\sqrt{k^*}) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} .$$

Therefore

$$\lim_{k^*\to\infty} u(r;n,\sigma,k^*) = \binom{n-1}{r-1} \lim_{k^*\to\infty} \Big\{ \int_0^{\infty} \{\Phi(x\sqrt{k^*})\}^{r-1} \{1-\Phi(x\sqrt{k^*})\}^{n-r} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

$$+ \int_{-\infty}^{0} \{\Phi(x\sqrt{k^*})\}^{r-1} \{1-\Phi(x\sqrt{k^*})\}^{n-r} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \Big\} .$$

By the dominated convergence theorem

$$\lim_{k^* \to \infty} u(n;n,\sigma,k^*) = \int_0^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = 1/2$$

$$\lim_{k^* \to \infty} u(1;n,\sigma,k^*) = \int_{-\infty}^{0} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = 1/2$$

and for any other $r = 2,\ldots,n-1$

$$\{\Phi(x\sqrt{k^*})\}^{r-1}\{1-\Phi(x\sqrt{k^*})\}^{n-r} \to 0 \quad \text{as} \quad k^* \to \infty\,,$$

so that, by the dominated convergence theorem,

$$\lim_{k^* \to \infty} u(r;n,\sigma,k^*) = \binom{n-1}{r-1}\int_{-\infty}^{\infty} 0\, \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = 0,\ r = 2,\ldots,n-1.$$

It thus appears that the spurious observation tends to occur at either $X_{(1)}$ or $X_{(n)}$ with equal probability (approaching 1/2) as the shape parameter of the spurious observation increases.

4.4 Estimation for Lognormal parameters

4.4.1 Standard Estimators

If $X \sim \Lambda(\mu, \sigma)$, M.L.E.'s of μ and σ may be obtained from M.L.E. of $W = \ln X$ since $W \sim N(\mu, \sigma^2)$. Thus the M.L.E.'s of μ and σ are .

$$\hat{\mu} = \frac{\sum_{i=1}^{n} W_{(i)}}{n} = \frac{\sum_{i=1}^{n} \ln X_{(i)}}{n}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (\ln X_{(i)})^2 - n\left(\frac{\sum_{i=1}^{n} \ln X_{(i)}}{n}\right)^2}{n}$$

Since $\hat{\mu}$ and $\hat{\sigma}^2$ are based on complete sufficient statistics, we can obtain best linear estimators $\hat{\mu}$ and $s^2 = \frac{n\hat{\sigma}^2}{n-1}$ and $\frac{n-1}{n+1} s^2$ has minimum MSE among estimators of σ^2 of the form ks^2. The BLIE estimators of μ and σ^2 are $\hat{\mu}$ and $\tilde{\sigma}^2 = \frac{n}{n+1} \hat{\sigma}^2$. Asymptotically, all three estimators are equivalent. However, using the property of MLE's that if $\hat{\gamma}$ is MLE of γ, $g(\hat{\gamma})$ is MLE of $g(\gamma)$, we may obtain M.L.E.'s of the mean and variance of the lognormal to be respectively, $e^{\hat{\mu} + \frac{1}{2}\hat{\sigma}^2}$ and $e^{2\hat{\mu}+\hat{\sigma}^2}(e^{\hat{\sigma}^2}-1)$.

To obtain confidence bounds for parameters of the two-parameter lognormal distributions, we use

$$\hat{\mu} = \bar{W} \sim N(\mu, \sigma^2/n).$$

Thus, for σ^2 known, a $(1-\alpha)$ 100% confidence interval (C.I.) for μ is given by

$$\bar{w} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{w} + z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} .$$

For σ^2 unknown, $\frac{n\hat{\sigma}^2}{\sigma^2} = \frac{\sum_{i=1}^{n} w_i^2 - nw^{-2}}{\sigma^2} \sim \chi^2_{n-1}$ df

and $\left(\frac{\bar{w}-\mu}{\sigma/\sqrt{n}}\right) \Big/ \sqrt{\frac{n\sigma^2}{\sigma^2(n-1)}} \sim t_{n-1}$ df

and a two-sided $(1-\alpha)$ 100% C.I. for μ is

$$\bar{w} + t_{\alpha/2,n-1df} \frac{\hat{\sigma}}{\sqrt{n-1}} \leq \mu \leq \bar{w} + t_{1-\alpha/2,n-1df} \frac{\hat{\sigma}}{\sqrt{n-1}} .$$

4.4.2 Estimators suggested for use

Case I: Scale change

Consider the estimator $\hat{\mu} = \frac{\sum_{i=1}^{n} \ell n\, X_i}{n}$. Under the homogeneous model, $E(\hat{\mu}) = \mu$ and $Var(\hat{\mu}) = \frac{\sigma^2}{n}$. Thus its $MSE(\hat{\mu}) = \frac{\sigma^2}{n}$. Under

the exchangeable model with (at most) one possible outlier,

$$E_{het}(\hat{\mu}) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{n-1}{n}\,\mu + \frac{1}{n}\,\mu_1\right)$$

$$= \mu\left\{1 + \frac{k^*-1}{n}\right\}$$

and

$$Var_{het}(\hat{\mu}) = \frac{1}{n^2} \sum_{i=1}^{n} Var_{het}\,(\ell n\, X_i)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \left(\frac{n-1}{n}\,\sigma^2 + \frac{1}{n}\,\sigma^2\right)$$

$$= \frac{\sigma^2}{n} \quad .$$

We then obtain $MSE_{het}(\hat{\mu}) = Var_{het}(\hat{\mu}) + \{Bias_{het}(\hat{\mu})\}^2$

$$= \frac{\sigma^2}{n} + \left\{\frac{\mu(k^*-1)}{n}\right\}^2$$

and, as $k^* \to \infty$, $MSE_{het}(\hat{\mu}) \to \infty$. Thus $\hat{\mu} = \dfrac{\sum_{i=1}^{n} \ell n\, X_i}{n}$ appears to be a poor estimator in this instance.

Assuming the exchangeable model with m outliers (generally $m \leq 10\%$ of the sample size), we have $n-m$ observations with p.d.f. $f(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp - \frac{1}{2\sigma^2}(\ell n\, x-\mu)^2$ and m observations with p.d.f. $f(x;\mu_1,\sigma)$, $\mu_1 \geq \mu$. The joint likelihood may be written as

$$L(\underline{x};\mu,\mu_1,\sigma,I) = \frac{1}{\binom{n}{m}} \left\{ \frac{\exp - \frac{1}{2\sigma^2} \sum_{x_i \notin I} (\ell n\, x_i-\mu)^2}{\prod_{x_i \notin I} \sigma x_i \sqrt{2\pi}} \right\} \left\{ \frac{\exp - \frac{1}{2\sigma^2} \sum_{x_i \in I} (\ell n\, x_i-\mu_1)^2}{\prod_{x_i \in I} \sigma x_i \sqrt{2\pi}} \right\}$$

We wish to obtain $\max_{\substack{I\in\mathcal{J} \\ \mu_1>\mu \\ \sigma>0}} L(\underline{x};\mu,\mu_1,\sigma,I)$ where $\mathcal{J}$ is the collection of all possible subsets of m outliers in a sample of size n. Kale (1974b) has shown for $\Psi(x) = \frac{dG(x)}{dF(x)}$ monotone increasing in x, $\hat{I} = \{x_{(n-m+1)},\ldots,x_{(n)}\}$ has maximum probability of being the set of spurious observations. For $\mu_1 > \mu$, $\Psi(x)$ is monotone increasing and hence

$$\max_{\substack{I\in\mathcal{J} \\ \mu_1>\mu \\ \sigma>0}} L(\underline{x};\mu,\mu_1,\sigma,I) = \max_{\substack{\mu_1>\mu \\ \sigma>0}} L(\underline{x};\mu,\mu_1,\sigma,\hat{I}) .$$

Now $K(\underline{x};\mu,\mu_1,\sigma,\hat{I}) = \ell n\, L(\underline{x};\mu,\mu_1,\sigma,\hat{I})$

$$= C - \frac{1}{2\sigma^2} \left\{ \sum_{i=1}^{n-m} (\ell n\, x_{(i)}-\mu)^2 + \sum_{i=n-m+1}^{n} (\ell n\, x_{(i)}-\mu_1)^2 \right\}$$

$$- n/2 \ln 2\pi\sigma^2 - \sum_{i=1}^{n} \ln x_i \quad .$$

Then $$\frac{\partial K}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n-m} (\ln x_{(i)} - \mu)$$

$$\frac{\partial K}{\partial \mu_1} = \frac{1}{\sigma^2} \sum_{i=n-m+1}^{n} (\ln x_{(i)} - \mu_1)$$

$$\frac{\partial K}{\partial \sigma^2} = \frac{1}{\sigma^4} \left\{ \sum_{i=1}^{n-m} (\ln x_{(i)} - \mu)^2 + \sum_{i=n-m+1}^{n} (\ln x_{(i)} - \mu)^2 \right\} - \frac{n}{\sigma^2}$$

and hence $$\hat{\mu}_{het} = \sum_{i=1}^{n-m} \frac{\ln x_{(i)}}{n-m}$$

$$\hat{\mu}_{1_{het}} = \frac{\sum_{i=n-m+1}^{n} \ln x_{(i)}}{m}$$

$$\hat{\sigma}^2_{het} = \frac{\sum_{i=1}^{n-m} (\ln x_{(i)} - \hat{\mu}_{het})^2 + \sum_{i=n-m+1}^{n} (\ln x_{(i)} - \hat{\mu}_{1_{het}})^2}{n}$$

Thus it appears that maximum likelihood estimation suggests trimmed means as estimators of μ_1 and μ ($\mu_1 > \mu$) and a pooled estimator, for σ^2, that utilizes these trimmed means.

Case II. Shape change

Under the homogeneous model, $E(\hat{\mu}) = E\left(\frac{\sum_{i=1}^{n} \ln x_i}{n}\right) = \mu$ and $Var(\hat{\mu}) = \frac{\sigma^2}{n}$. Under the exchangeable model with at most one possible outlier,

$$E_{het}(\hat{\mu}) = E\left(\frac{\sum_{i=1}^{n} \ln x_i}{n}\right) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{n-1}{n}\mu + \frac{1}{n}\mu\right) = \mu$$

and

$$Var_{het}(\hat{\mu}) = Var\left(\frac{\sum_{i=1}^{n} \ln x_i}{n}\right) = \frac{1}{n^2} \sum_{i=1}^{n} Var_{het}(\ln x_i)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \left(\frac{n-1}{n}\sigma^2 + \frac{1}{n}\sigma_1^2\right)$$

$$= \frac{\sigma^2(n-1+k^*)}{n^2} .$$

Thus $MSE_{het}(\hat{\mu}) = Var_{het}(\hat{\mu}) + \{Bias_{het}(\hat{\mu})\}^2$

$$= \frac{\sigma^2(n-1+k^*)}{n^2}$$

which tends to infinity as $k^* \to \infty$. Thus $\hat{\mu}$ appears to be a poor estimator of μ.

Since in this instance the spurious observation tends to appear at $X_{(1)}$ or at $X_{(n)}$, an estimator of μ based on the A-rule, W-rule, or S-rule (see Appendix I) would appear preferable to $\hat{\mu} = \frac{\sum_{i=1}^{n} \ln x_i}{n}$. Similar estimators for σ^2 appear preferable to $s^2_{\ln x}$.

CHAPTER V

The Weibull Distribution

We shall show that for i.i.d.r.v.'s the family of Weibull distributions, indexed by the shape parameter η, is outlier-prone completely on the right. On the other hand, we shall show that the exchangeable model based on the Weibull family of distributions indexed by the shape parameter is outlier-resistant completely on the right. For this exchangeable model, we shall determine which observation is most likely to be the spurious one and find asymptotic limits for these probabilities as heterogeneity increases. Also we shall examine the problem of estimation in the presence of outliers.

5.1 Characteristics of the Weibull Distribution

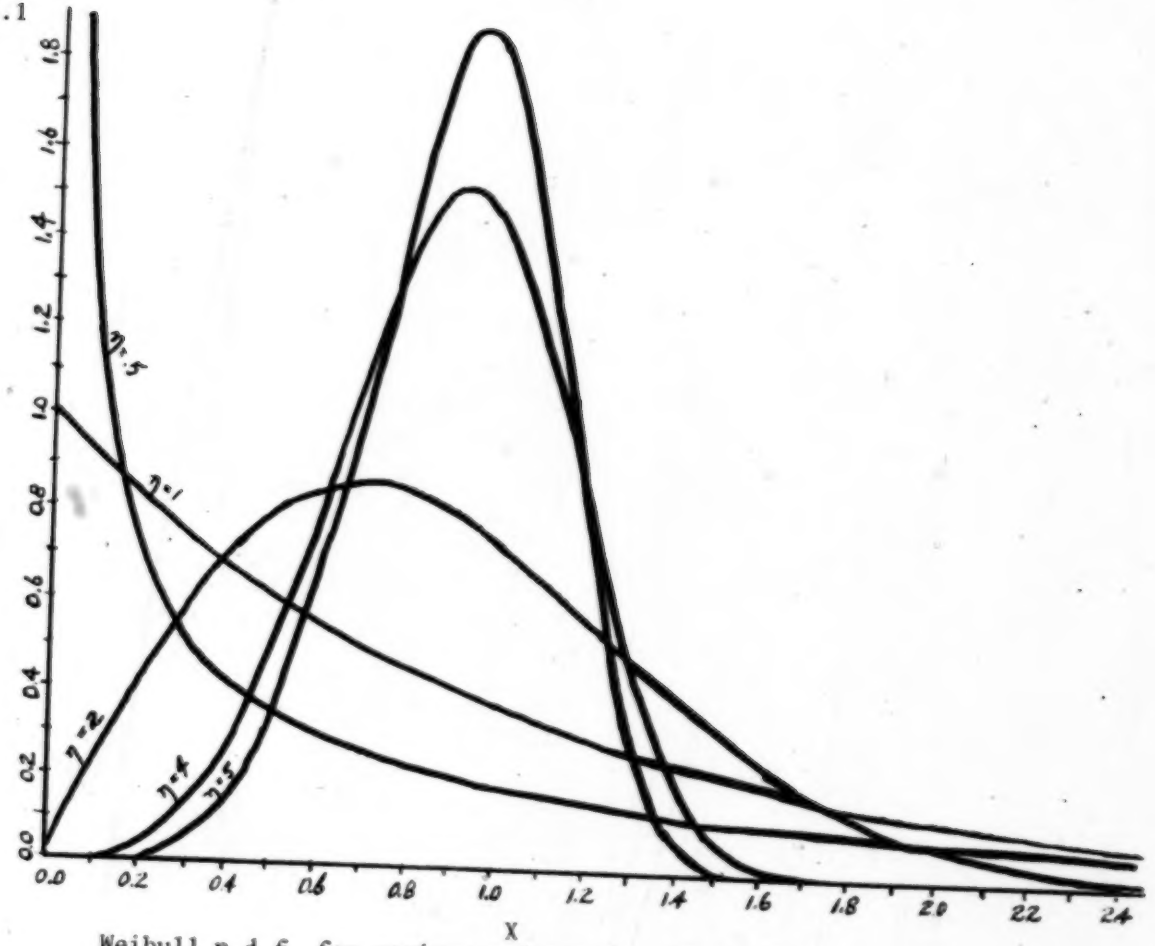
The random variable X is said to have the three-parameter Weibull distribution if the probability density function of X is

$$f(x;\theta,\eta,\tau) = \begin{cases} \frac{\eta}{\theta}\left(\frac{x-\tau}{\theta}\right)^{\eta-1} \exp -\left(\frac{x-\tau}{\theta}\right)^{\eta}, & x > \tau,\ \eta > 0,\ \theta > 0,\ \tau \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

giving a d.f. of $F(x;\theta,\eta,\tau) = 1 - \exp\left\{-\left(\frac{x-\tau}{\theta}\right)^{\eta}\right\}$, and hazard rate (intensity) of $h(x;\theta,\eta,\tau) = \frac{\eta(x-\tau)^{\eta-1}}{\theta^{\eta}}$. In this case we write $X \sim \text{WEI}(\theta,\eta,\tau)$.

Figure 5.1.1 illustrates the shape of the probability density function $f(x;1,\eta,0)$ for various values of η.

Fig.5.1.1



Weibull p.d.f. for various values of η ($\theta=1$, $\tau=0$)

The Weibull distribution has many applications in life-testing or in problems where a skewed distribution is required. It may be considered to be a generalization of the gamma distribution to allow the hazard rate to depend on a power of X. In contrast to the exponential distribution with constant hazard (failure) rate $h(x;\theta) = \frac{1}{\theta}$ (i.e. $\eta = 1$), the Weibull distribution allows decreasing hazard rates (i.e. $\eta < 1$ implies work-hardened materials) or increasing hazard rates (i.e. $\eta > 1$ implies wearout). Many items, especially nonelectronic parts, exhibit increasing failure rates. Leiblein and Zelen (1956) used it to model ball-bearing failures; Kao (1959) used it to model vacuum-tube failure. Weibull (1951) derived it in the analysis of breaking strengths. Whatever the underlying distribution for positive random variables, the Weibull distribution is one of only two possible limit laws for $\min(X_1,\ldots,X_n)$ as $n \to \infty$. As such it is used to model metal fatigue breaking strength where strength is that at the weakest flaw (e.g. breaking strength of chain, ceramics, lumber, concrete, aircraft parts, etc.). In many applications the location parameter τ is known and without loss of generality we may take $\tau = 0$.

There exists a relationship between $WEI(\theta,\eta,0)$ and the Extreme-Value distribution $EV_I(\xi,b)$ given by $F(y) = 1-\exp\{-\frac{\exp(y-\xi)}{b}\}$, $-\infty < y < \infty$, $-\infty < \xi < \infty$, $b > 0$. If $X \sim WEI(\theta,\eta,0)$ then

$Y = \ln X \sim EV_I(\xi = \ln \theta,\ b = 1/\eta)$. Thus the Weibull distribution competing with the lognormal distribution is similar to the EV_I distribution competing with the normal distribution. For $Y \sim EV_I(\xi,b)$, $E(Y) = \xi - b\gamma$ and $Var(Y) = \frac{\pi^2 b^2}{6}$ where $\gamma = .5772$. (Euler's constant).

For known shape parameter η, we may use the transformation $Y = X^\eta$ to obtain $Y \sim EXP(\theta^\eta) = GAM(\theta^\eta,1,0)$. Outliers here may be handled using methods for the single parameter exponential family (see Chapter II).

5.2. Outlier-proneness of the Weibull and related models

First, let us restrict ourselves to the standard subfamily indexed only by shape parameter η. The probability density function is

$$f(x;\eta) = \begin{cases} \eta x^{\eta-1} e^{-x^{\eta}} & , \ x > 0 , \ \eta > 0 \\ 0 & , \ \text{otherwise} \end{cases} .$$

The corresponding distribution function (d.f.) will be denoted as $F(x;\eta)$ and the family of such distribution functions as $\mathcal{F}$. In this model we consider i.i.d. random variables $X_1,\ldots,X_n$ with order statistics $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$.

Theorem 5.2.1: For i.i.d. random variables the family of Weibull distributions is outlier-prone on the right.

Proof; In order to prove Theorem 5.2.1 we notice that

$$P\{X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)}) \cap X_{(1)} > 0\}$$

$$= P\{X_{(1)} > 0\} \cdot P\{X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)}) | X_{(1)} > 0\}$$

$$= P\{X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)})\} P\{X_{(1)} > 0 | X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)})\} .$$

Now $P\{X_{(1)} > 0\} = 1$ and

$$P\{X_{(1)} > 0 | X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)})\} = 1$$

so that, following Neyman and Scott (1971),

$$P(k,n|\eta) = P\{X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)})\} = P\{X_{(n)} > X_{(n-1)} + k(X_{(n-1)}-X_{(1)}) \mid X_{(1)} > 0\}$$

$$> P\{X_{(n)} > (k+1)X_{(n-1)}\} = Q(k,n|\eta) .$$

Thus it is sufficient to prove the stronger assertion that, as $\eta \to 0$, $Q(k,n|\eta) \to 1$.

$$Q(k,n|\eta) = n \int_0^\infty F^{n-1}\left(\frac{x}{k+1};\eta\right) f(x;\eta)dx$$

$$= n\int_0^\infty \left\{1 - e^{-\left(\frac{x}{k+1}\right)^\eta}\right\}^{n-1} \eta x^{\eta-1} e^{-x^\eta} dx .$$

Setting $u = 1 - e^{-\left(\frac{x}{k+1}\right)^\eta}$ we obtain

$$Q(k,n|\eta) = n(k+1)^\eta \int_0^1 u^{n-1}(1-u)^{(k+1)^\eta - 1} du$$

$$= \frac{n\Gamma(n)\Gamma\{(k+n)^\eta\}(k+1)^\eta}{\Gamma\{n+(k+1)^\eta\}} .$$

It then follows that, as $\eta \to 0$, $Q(k,n|\eta) \to 1$ and thus $P(k,n|\eta) \to 1$ and $\Pi_1(k,n|\mathcal{F}) = 1$. Since this result holds for all $k > 0$, $n > 2$, the family of Weibull distributions is outlier-prone on the right.

Consider now the exchangeable model with (at most) one outlier observation. Kale (1975b) has shown that the exchangeable model based on scale parameter families for non-negative random variables with a possible change in scale is outlier-prone completely on the right. This would apply to the exchangeable model based on the Weibull distribution with possible change in the scale parameter θ. Let us consider the other possibility - the exchangeable model involving the Weibull distribution with possible change in the shape parameter η. We shall show that this model is outlier-resistant completely on the right.

Consider the situation where $n-1$ observations are from $WEI(1,\eta,0)$ and one observation is from $WEI(1,k^*\eta,0)$, $0 < k^* \leq 1$ and, a priori, each observation is equally likely to be the spurious one. Then $f(x;\underline{\theta})$ is $WEI(1,\eta,0)$ and $f(x;\underline{\xi})$ is $WEI(1,k^*\eta,0)$ and the likelihood may be written as

$$L(\underline{x};\eta,k^*) = \frac{1}{n}\sum_{r=1}^{n}\prod_{i\neq r} f(x_i;\underline{\theta})f(x_r;\underline{\xi}),\ x_i > 0,\ 0 < k^* \leq 1$$

($k^* \leq 1$ since we are considering $X_{(1)}$ as a possible outlier). Then the joint density of the order statistics may be written as

$$f(x_{(1)},\ldots,x_{(n)}) = \frac{1}{n}\cdot n!\sum_{r=1}^{n}\frac{f(x_r;\eta k^*)}{f(x_r;\eta)}\prod_{i=1}^{n} f(x_i;\eta)$$

and

$$g(x_{(1)},x_{(n-1)},x_{(n)})$$

$$= \int_{x_{(1)}}^{x_{(n-1)}} \int_{x_{(1)}}^{x_{(n-2)}} \cdots \int_{x_{(1)}}^{x_{(3)}} f(x_{(1)},x_{(2)},\ldots,x_{(n)})dx_{(2)}\cdots dx_{(n-2)}$$

$$= (n-1)! \sum_{r=1}^{n} h_r(x_{(1)},x_{(n-1)},x_{(n)})$$

where

$$h_r(x_{(1)},x_{(n-1)},x_{(n)}) = \int_{S_{n-3}} \frac{f(x_r;\eta_1)}{f(x_r;\eta)} \prod_{i=1}^{n} f(x_i;\eta)dx_{(2)}\cdots dx_{(n-2)}$$

and S_{n-3} is the region $x_{(1)} < x_{(2)} < \ldots < x_{(n-2)} < x_{(n-1)}$.

For $r = 2,\ldots,n-2$

$$h_r(x_{(1)},x_{(n-1)},x_{(n)}) = \frac{f(x_{(1)};\eta)f(x_{(n-1)};\eta)f(x_{(n)};\eta)}{(r-2)!(n-r-2)!}$$

$$\int_{x_{(1)}}^{x_{(n-1)}} \{F(x_{(r)};\eta)-F(x_{(1)};\eta)\}^{r-2}\{F(x_{(n-1)};\eta)-F(x_{(r)};\eta)\}^{n-r-2}$$

$$f(x_{(r)};\eta_1)dx_{(r)}$$

while

$h_1(x_{(1)},x_{(n-1)},x_{(n)})$

$$= f(x_{(1)};\eta_1)f(x_{(n-1)};\eta)f(x_{(n)};\eta)\ \frac{[F(x_{(n-1)};\eta)-F(x_{(1)};\eta)]^{n-3}}{(n-3)!}$$

and

$h_{n-1}(x_{(1)},x_{(n-1)},x_{(n)})$

$$= f(x_{(1)};\eta)f(x_{(n-1)};\eta_1)f(x_{(n)};\eta)\ \frac{[F(x_{(n-1)};\eta)-F(x_{(1)};\eta)]^{n-3}}{(n-3)!}$$

and

$h_n(x_{(1)},x_{(n-1)},x_{(n)})$

$$= f(x_{(1)};\eta)f(x_{(n-1)};\eta)f(x_{(n)};\eta_1)\ \frac{[F(x_{(n-1)};\eta)-F(x_{(1)};\eta)]^{n-3}}{(n-3)!}$$

Letting $t(y) = f(y;\eta)$ and $s(y) = f(y;\eta_1)$ we obtain again expression (4.2.2) and, setting

$$p = P\{X_{(r)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

$$= \int_0^{\infty}\int_0^{x_{(n-1)}}\int_{(k+1)x_{(n-1)}-kx_{(1)}}^{\infty} g(x_{(1)},x_{(n-1)},x_{(n)})dx_{(n)}dx_{(1)}dx_{(n-1)}$$

and $x = x_{(1)}$ and $y = x_{(n-1)}$, from Theorem 4.2.1

$$p \leq 1 - k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

where S and s are, respectively, the distribution function and probability density function of the spurious observation and T and t are, respectively, the distribution function and probability density function of the non-spurious observations.

Theorem 5.2.2: The exchangeable model with (at most) one spurious observation based on the Weibull family of distributions indexed by shape parameter η is outlier-resistant completely on the right.

Proof: From Theorem 4.2.1 we know that if

$$p = P\{X_{(r)} > (k+1)X_{(n-1)} - kX_{(1)}\}$$

then, for non-negative random variables

$$p \leq 1 - k(n-1)(n-2)\int_0^\infty t(y)\int_0^y S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

where $x = x_{(1)}$, $y = x_{(n-1)}$, S and s are, respectively, the distribution function and p.d.f. of a spurious observation and T and t are, respectively, the distribution function and p.d.f. of a non-spurious observation.

If, for all $k > 0$, $n > 2$ $\sup p < 1$ then the family is outlier-resistant completely on the right. It is then sufficient to show that

$$(5.2.1)\quad J = k(n-1)(n-2)\int_0^{\infty} t(y)\int_0^{y} S(x)\{T(y)-T(x)\}^{n-3}t\{(k+1)y-kx\}dxdy$$

$$> 0.$$

Now

$$t(y) = \begin{cases} \eta y^{\eta-1}e^{-y^{\eta}}, & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$s(x) = \begin{cases} \eta_1 x^{\eta_1-1}e^{-x^{\eta_1}}, & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$T(b) = \begin{cases} 1-e^{-b^{\eta}}, & b > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$S(b) = \begin{cases} 1-e^{-b^{\eta_1}}, & b > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Then the left hand side of expression (5.2.1) becomes

$$(5.2.2)\quad J = k(n-1)(n-2)\int_0^{\infty} \eta y^{\eta-1}e^{-y^{\eta}}\int_0^{y}(1-e^{-x^{\eta_1}})\{(1-e^{-y^{\eta}})-(1-e^{-x^{\eta}})\}^{n-3}$$

(continued)

$$\times \eta\{(k+1)y-kx\}^{\eta-1} e^{-[(k+1)y-kx]^{\eta}} dxdy.$$

$$J \geq k(n-1)(n-2)\int_3^5 \eta y^{\eta-1} e^{-y^{\eta}} \int_1^2 (1-e^{-x^{\eta_1}})\{(1-e^{-y^{\eta}})-(1-e^{-x^{\eta}})\}^{n-3}$$

$$\eta\{(k+1)y-kx\}^{\eta-1} e^{-[(k+1)y-kx]^{\eta}} dxdy.$$

Now for $\eta > 0$, $k > 0$, $\eta_1 > 0$, $3 \leq y \leq 5$, $1 \leq x \leq 2$

$$y^{\eta-1} = \frac{y^{\eta}}{y} \geq \frac{3^{\eta}}{5} > 0$$

$$e^{-y^{\eta}} \geq e^{-5^{\eta}} > 0$$

$$(1-e^{-x^{\eta_1}}) \geq (1-e^{-1^{\eta_1}}) = 1 - \frac{1}{e} > 0 \quad \text{independently of} \quad \eta_1 > 0.$$

$$\{(1-e^{-y^{\eta}}) - (1-e^{-x^{\eta}})\}^{n-3} = \{e^{-x^{\eta}} - e^{-y^{\eta}}\}^{n-3} \geq \{e^{-2^{\eta}} - e^{-3^{\eta}}\}^{n-3} > 0$$

$$\{(k+1)y - kx\}^{\eta-1} = \frac{\{(k+1)y-kx\}^{\eta}}{\{(k+1)y-kx\}} \geq \frac{\{(k+1)3-2k\}^{\eta}}{\{(k+1)5-k\}} = \frac{(k+3)^{\eta}}{\{4k+5\}} > 0$$

$$e^{-[(k+1)y-kx]^{\eta}} \geq e^{-[(k+1)5-k]^{\eta}} = e^{-[4k+5]^{\eta}} > 0 .$$

Therefore

$$J > k\eta^2(n-1)(n-2)\,\frac{3^{\eta}}{5}\,e^{-5^{\eta}}\left(1-\frac{1}{e}\right)\{e^{-2^{\eta}}-e^{-3^{\eta}}\}^{n-3}\,\frac{(k+3)^{\eta}}{4k+5} \times e^{-[4k+5]^{\eta}} 2$$

> 0 for all $k > 0$, $n > 2$, $\eta, \eta_1 > 0$.

Thus the exchangeable model with (at most) one spurious observation based on the Weibull family of distributions indexed by shape parameter η is outlier-resistant completely on the right.

5.3 Detection of Outliers

Consider a situation where $n-1$ observations are distributed as $WEI(1,\eta,0)$ and one is distributed as $WEI(1,k^*\eta,0)$, $0 < k \leq 1$ and we have the exchangeable model. If $k^* < 1$ we have exactly one spurious observation. Then $\Psi(x) = \frac{dG}{dF} = k^* x^{k^*\eta-\eta} e^{-(x^{k^*\eta}-x^{\eta})}$ is not monotone. Letting $u(r;n,\eta,k^*) = P(X_{(r)}$ is the spurious observation), we have

$$u(r;n,\eta,k^*) = \binom{n-1}{r-1} \int_0^{\infty} (1-e^{-x^{\eta}})^{r-1} (e^{-x^{\eta}})^{n-r} k^*\eta x^{k^*\eta-1} e^{-x^{k^*\eta}} dx$$

and, setting $y = e^{-x^{k^*\eta}}$, we obtain

$$u(r;n,k^*) = \binom{n-1}{r-1} \int_0^{1} (1-e^{-(\ell n\ 1/y)^{1/k^*}})^{r-1} (e^{-(\ell n\ 1/y)^{1/k^*}})^{n-r} dy$$

which is independent of η. Therefore, without loss of generality (w.l.o.g.), we may take $\eta = 1/k^*$, giving

$$u(r;n,k^*) = \binom{n-1}{r-1} \int_0^{\infty} (1-e^{-x^{1/k^*}})^{r-1} (e^{-x^{1/k^*}})^{n-r} e^{-x} dx$$

and in particular

$$u(1;n,k^*) = \int_0^{\infty} (e^{-x^{1/k^*}})^{n-1} e^{-x} dx$$

$$= \frac{1}{(n-1)^{k^*}} \int_0^\infty e^{-v^{1/k^*}} e^{-v/(n-1)^{k^*}} dv$$

$$= \frac{1}{(n-1)^{k^*}} L\left(u = 1/k^*,\ p = \frac{1}{(n-1)^{k^*}}\right)$$

where $L(u,p) = \int_0^\infty e^{-v^k} e^{-pv} dv$ is the Laplace transform of e^{-v^u} (Re $u > 0$).

Now $u(1;n,k^*) = \int_0^\infty \left(e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx \quad , \quad n > 1$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty x^m e^{-(n-1)x^{1/k^*}} dx$$

and setting $t = (n-1)x^{1/k^*}$ we obtain

$$u(1;n,k^*) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{t=0}^{\infty} \frac{e^{-t} k^* t^{k^*(m+1)-1}}{(n-1)^{k^*(m+1)}} dt$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m k^* \Gamma\{k^*(m+1)\}}{m!(n-1)^{k^*(m+1)}} , \quad n > 1$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-1)^{k^*(m+1)}} , \quad n > 1 \qquad (5.3.1)$$

For computational accuracy to the third decimal place, we would require

q terms where q is chosen such that $\frac{\Gamma\{k^*(q+1)+1\}}{(q+1)!(n-1)^{k^*(q+1)}} \leq 0.0005$. (Apostol 1964). For fixed n, as $k^* \to \infty$ (heterogeneity increases) q increases; as $k^* \to 1$ (heterogeneity decreases) q decreases. Table 5.3.1 shows q values for some selected (n,k^*) values.

Table 5.3.1 Number of terms required to calculate $u(1;n,k^*)$ accurate to third decimal

		Sample size n			
		5	10	20	50
Coefficient of spuriosity k^*	1/4	5	5	4	4
	1/2	5	4	3	3
	3/4	5	3	3	2
	1	5	3	2	1

We can also develop a recursive formula for $u(r;n,k^*)$ in terms of $u(1;j,k^*)$, $j \leq n$. From equation 5.3.1 we have

$$u(1;n,k^*) = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-1)^{k^*(m+1)}} \quad , \quad n > 1$$

and, for the case of $n = 1$, we may define $u(1;1,k^*) = 1$. For the special case of $r = n$ we may write

$$u(n;n,k^*) = \int_0^{\infty} \left(1-e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx$$

$$= \int_0^\infty \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j (e^{-x^{1/k^*}})^j e^{-x} dx$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \int_0^\infty e^{-jx^{1/k^*}} e^{-x} dx$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty x^m e^{-jx^{1/k^*}} dx$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{k^* \Gamma\{k^*(m+1)\}}{j^{k^*(m+1)}}$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} \frac{\Gamma\{k^*(m+1)+1\}}{j^{k^*(m+1)}}$$

$$= 1 + \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j u(1;j+1,k^*)$$

$$= \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j u(1;j+1,k^*) \quad \text{where} \quad u(1;1,k^*) = 1 \quad \text{by}$$

definition.

For $r = 2,3,\ldots,n-1$

$$u(r;n,k^*) = \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j \int_0^\infty (e^{-x^{1/k^*}})^{n-r+j} e^{-x} dx$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty x^m e^{-x^{1/k^*}(n-r+j)} dx$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \left[\sum_{n=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-r+j)^{k^*(m+1)}} \right]$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j u(1;n-r+j+1,k^*) \ .$$

Thus we may write

$$u(r;n,k^*) = \begin{cases} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-1)^{k^*(m+1)}} & \text{for } r = 1,\ n \geq 2 \\ 1 \quad \text{if} \quad r = 1,\ n = 1 \\ \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j u(1;n-r+j+1,k^*) & \text{for } r = 2,\ldots,n \\ 0 \qquad \text{otherwise} \end{cases}$$

(See Appendix IV). Consider now

$$\lim_{k^* \to 1} u(r;n,k^*) = \begin{cases} \int_0^{\infty} \lim_{k^* \to 1} \left(e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx \ , & r = 1 \\ \int_0^{\infty} \lim_{k^* \to 1} \left(1-e^{-x^{1/k^*}}\right)^{n-1} e^{-x} dx \ , & r = n \\ \binom{n-1}{r-1} \int_0^{\infty} \lim_{k^* \to 1} \left(1-e^{-x^{1/k^*}}\right)^{r-1} \left(e^{-x^{1/k^*}}\right)^{n-r} e^{-x} dx \ , & \\ \qquad\qquad\qquad\qquad\qquad\qquad r = 2,\ldots,n-1 & \\ 0 \qquad\qquad , \quad \text{otherwise} & \end{cases}$$

Thus we find that

$$\lim_{k^*\to 1} u(r;n,k^*) = \begin{cases} \int_0^\infty e^{-xn}dx & , r = 1 \\ \int_0^\infty (1-e^{-x})^{n-1}e^{-x}dx & , r = n \\ \binom{n-1}{r-1}\int_0^\infty (1-e^{-x})^{r-1}e^{-x(n-r+1)}dx & , r = 2,\ldots,n-1 \\ 0 & , \text{otherwise} \end{cases}$$

$$= \begin{cases} 1/n & , r = 1 \\ 1/n & , r = n \\ \binom{n-1}{r-1}\frac{\Gamma(r)\Gamma(n-r+1)}{\Gamma(n+1)} = 1/n & , r = 2,3,\ldots,n-1 \\ 0 & , \text{otherwise} \end{cases}.$$

The interpretation of this result is that, as $k^* \to 1$ ($k^* = 1$ is the case of homogeneous data), the heterogeneity of the data is decreasing and the probability of any order statistic being spurious approaches $1/n$, i.e. all equally likely.

Now consider $\lim_{k^*\to 0} u(r;n,k^*)$. We have

$$\lim_{k^*\to 0} u(r;n,k^*) = \begin{cases} \int_0^\infty \lim_{k^*\to 0} (e^{-x^{1/k^*}})^{n-1}e^{-x}dx & , r = 1 \\ \int_0^\infty \lim_{k^*\to 0} (1-e^{-x^{1/k^*}})^{n-1}e^{-x}dx & , r = n \\ \binom{n-1}{r-1}\int_0^\infty \lim_{k^*\to 0} (1-e^{-x^{1/k^*}})^{r-1}(e^{-x^{1/k^*}})^{n-r}e^{-x}dx & , \\ & r = 2,\ldots,n-1 \\ 0 & , \text{otherwise} \end{cases}$$

Now

$$\lim_{k^*\to 0} u(1;n,k^*) = \lim_{k^*\to 0} \int_0^\infty (e^{-x^{1/k^*}})^{n-1} e^{-x}dx$$

$$= \lim_{k^*\to 0} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty x^m (e^{-x^{1/k^*}})^{n-1} dx$$

$$= \lim_{k^*\to 0} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)\}}{m!\left|\frac{1}{k^*}\right|(n-1)^{k^*(m+1)}} \quad , \quad n > 1$$

$$= \lim_{k^*\to 0} \sum_{m=0}^{\infty} \frac{(-1)^m k^*(m+1)\Gamma\{k^*(m+1)\}}{(m+1)!(n-1)^{k^*(m+1)}}$$

$$= \lim_{k^*\to 0} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(n-1)^{k^*(m+1)}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} = \frac{(-1)^0}{1!} + \frac{(-1)^1}{2!} + \frac{(-1)^3}{3!} + \cdots$$

$$= 1 - \frac{1}{e} \doteq .63 .$$

For $r = n$

$$\lim_{k^*\to 0} u(n;n,k^*) = \lim_{k^*\to 0} \int_0^\infty (1-e^{-x^{1/k^*}})^{n-1} e^{-x}dx$$

$$= \lim_{k^*\to 0} \int_0^\infty \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j (e^{-x^{1/k^*}})^j e^{-x}dx$$

$$= \lim_{k^*\to 0} \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j \int_0^\infty e^{-jx^{1/k^*}} e^{-x} dx$$

$$= \left\{ \lim_{k^*\to 0} \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \int_0^\infty e^{-jx^{1/k^*}} e^{-x} dx \right\} + \int_0^\infty e^{-x} dx$$

$$= \left\{ \lim_{k^*\to 0} \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \left[\sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!\, j^{k^*(m+1)}} \right] \right\} + 1$$

$$= \left\{ \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \left[\sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(1)}{(m+1)!} \right] \right\} + 1$$

$$= \left\{ \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \left(1 - \frac{1}{e}\right) \right\} + 1$$

$$= \left(1 - \frac{1}{e}\right) \left\{ \sum_{j=1}^{n-1} \binom{n-1}{j}(-1)^j \right\} + 1$$

$$= \left(1 - \frac{1}{e}\right) \left\{ \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j - \binom{n-1}{0}(-1)^0 \right\} + 1$$

$$= \left(1 - \frac{1}{e}\right)(-1) + 1$$

$$= \frac{1}{e} \doteq .37 \quad .$$

For $r = 2,3,\ldots,n-1$

$$\lim_{k^*\to 0} u(r;n,k^*) = \lim_{k^*\to 0} \binom{n-1}{r-1} \int_0^\infty \left(1-e^{-x^{1/k^*}}\right)^{r-1} \left(e^{-x^{1/k^*}}\right)^{n-r} e^{-x}\,dx$$

$$= \lim_{k^*\to 0} \binom{n-1}{r-1} \int_0^\infty \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j \left(e^{-x^{1/k^*}}\right)^{j+n-r} e^{-x}\,dx$$

$$= \lim_{k^*\to 0} \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j$$

$$\left\{ \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^\infty x^m e^{-(j+n-r)x^{1/k^*}}\,dx \right\}$$

$$= \lim_{k^*\to 0} \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j$$

$$\left\{ \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)\}}{m!\left|\frac{1}{k^*}\right|(j+n-r)^{k^*(m+1)}} \right\}$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j \lim_{k^*\to 0} \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\{k^*(m+1)+1\}}{(m+1)!(j+n-r)^{k^*(m+1)}} \right\}$$

$$= \binom{n-1}{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} \right\}$$

$$= \binom{n-1}{r-1}\left(1-\frac{1}{e}\right)\left\{ \sum_{j=0}^{r-1} \binom{r-1}{j}(-1)^j \right\}$$

$$= \binom{n-1}{r-1}\left(1-\frac{1}{e}\right)(-1+1)$$

$= 0$.

Thus, as $k^* \to 0$ (i.e. one observation with Weibull shape parameter much smaller than the rest), we have

$$\lim_{k^* \to 0} u(r;n,k^*) = \begin{cases} 1 - \frac{1}{e} & , \quad r=1 \\ \frac{1}{e} & , \quad r=n \\ 0 & , \quad \text{otherwise} \end{cases}$$

In this situation, heterogeneity is more evident and the probability that the first order statistic is spurious approaches $1 - \frac{1}{e} = .63$, the probability that the last order statistic is spurious approaches $\frac{1}{e} = .37$, and the probability that any other statistic is spurious approaches zero.

5.4 Estimation for Weibull parameters

5.4.1 Standard Estimators

Complete sufficient statistics do not exist for the Weibull distribution. Much of the estimation for the Weibull distribution falls into one of three categories:

i) For M.L.E's, the distributional results are not mathematically tractable hence Monte Carlo simulation is needed to tabulate percentage points. While the M.L.E.'s have optimality properties as n increases, they are not in closed form and require computer solution to obtain

$$\hat{\eta} = \left\{ \frac{\sum_{i=1}^{n} X_i^{\eta} \ln x_i}{\sum_{i=1}^{n} X_i^{\eta}} - \frac{1}{n} \sum_{i=1}^{n} \ln X_i \right\}^{-1} \quad \text{and} \quad \hat{\theta}^{\hat{\eta}} = \frac{\sum_{i=1}^{n} X_i^{\hat{\eta}}}{n} .$$

For $n \geq 100$, $\hat{\eta}$ is asymptotically $N(\eta, \frac{.608\eta^2}{n})$ [see Cohen (1965) and Harter and Moore (1965)]. Inference procedures based on M.L.E.'s depend on pivotal quantities $\frac{\hat{\eta}}{\eta}$ and $\hat{\eta} \ln(\frac{\hat{\theta}}{\theta})$ [see Thoman, Bain and Antle (1970)].

ii) Best Linear Estimators have properties similar to M.L.E.'s. Mann and Fertig (1973) considered BLIE for the parameter b of EV_I. The BLIE $\tilde{b}_{BLIE} = \sum_{i=1}^{r} c_{i,r,n} X_{(i)}$ is a linear function of the BLUE and a maximal invariant since $\sum_{i=1}^{r} c_{i,r,n} = 0$. ($r$ is the size of the

censored sample). The distribution of $\frac{\tilde{b}_{BLIE}}{b}$ is independent of b and ξ. Mann and Fertig (1977) also took Hassanein's (1972) asymptotically unbiased optimum estimators $\hat{b}_k = \sum_{i=1}^{k} b_{i,k} X_{(n_i)}$, $n_i = [\gamma_{i,k} n] + 1$ and obtained, for small samples, the unbiased

$$b^* = \frac{\hat{b}_k}{\bar{b}_{k,n}}$$

where $\bar{b}_{k,n} = E[\sum_{i=1}^{k} b_{i,k} Z_{(n_i)}]$, where $Z_i \sim EV_I(0,1)$ and $\frac{2b^*}{cb} \doteq \chi^2_{2/c}df$ and the BLIE $b^*_{BLIE} = \frac{b^*}{1+c_{k,n}}$ where $c_{k,n} = Var(\frac{b^*}{b})$.

iii) Simple estimators, such as good linear unbiased estimators (GLUE's) and modified GLUE's, are identical to BLUE's if $r = 2$ and similar but not identical if $r > 2$, where r is the size of the censored sample. They are essentially equivalent to M.L.E.'s for censored sampling but are slightly less efficient for the complete sample case. Bain (1973) suggested using the approximate BLUE

$$\hat{b}_B = \frac{\sum_{i=1}^{r} |y_{(i)} - y_{(r)}|}{nk_{r,n}} = \frac{(r-1)y_{(r)} - \sum_{i=1}^{r-1} y_{(i)}}{nk_{r,n}}$$

where $y_i \sim EV_I(\xi,b)$ and $k_{r,n} = -\frac{1}{n} E\{\sum_{i=1}^{n-1} (w_i - w_r)\}$ where w_i are

the order statistics of $EV_I(0,1)$. Then $2nk_{r,n}\frac{\hat{b}_B}{b} \sim \chi^2_{2nk_{r,n}df}$ (if $r/n < 1/2$) and $\hat{\eta}_B = \frac{nk_{r,n}^{-1}}{nk_{r,n}\hat{b}_B}$ is approximately unbiased for η. For complete samples (i.e. $r = n$), $\hat{b}_B$ has zero asymptotic relative efficiency.

Englehardt and Bain (1973) suggested $\hat{b}_s = \frac{\sum_{i=1}^{r} |y_{(i)}-y_{(s)}|}{nk_{s,r,n}}$ as an improvement where

$$s = \begin{cases} n & 2 \le n \le 15 \\ n-1 & 16 \le n \le 24 \\ [.892\ n] + 1\ , & n \ge 25 \\ r\ , & r \le .9n \end{cases} \quad \text{and} \quad r > .9n$$

and $h\frac{\hat{b}_s}{b} \overset{\cdot}{\sim} \chi^2_{h\ df}$ where $h = \frac{2}{Var(\frac{\hat{b}_s}{b})}$.

For $\frac{r}{n} \to 0$ (very heavy censoring) as $n \to \infty$, $\hat{b}_B$ and M.L.E. $\hat{b}$ appear to agree. Compared to $\hat{b}_B$, $\hat{b}_s$ has relative efficiency ranging from 1 (n=2) to 0.82 (n=25).

Mann and Fertig (1975) converted Bain's estimator to an approximate BLIE. For $n \ge 20$, $\frac{2\frac{\hat{b}_s}{b}}{\ell_{r,n}} \overset{\cdot}{\sim} \chi^2_{2/\ell_{r,n}df}$ where $\hat{b}_s = \frac{\sum_{i=1}^{r} |y_{(s)}-y_{(i)}|}{nk_{s,r,n}}$ and approximate BLIE $\tilde{b}_{MF} = \frac{\hat{b}_s}{1+\ell_{r,n}}$, where

$\ell_{r,n} = \mathrm{Var}\left(\frac{\hat{b}_s}{b}\right) = \frac{1}{nk_{s,r,n}}$. Then $\mathrm{MSE}(\tilde{b}_{MF}) = \frac{\ell_{r,n} b^2}{1+\ell_{r,n}}$. For heavy censoring ($r \ll n$), $\tilde{b}_{MF}$ is closer to $\hat{b}_{MLE}$ then $\hat{b}_s$.

In terms of the Weibull shape parameter η, $a\hat{\eta} = a\left(\frac{1}{\tilde{b}_{MF}}\right)$ is unbiased for η if $a = \frac{h-2}{h+2}$ and $d\hat{\eta} = d\left(\frac{1}{\tilde{b}_{MF}}\right)$ has minimum MSE if $d = \frac{h-4}{h-2}$ where $\frac{h+2}{h} = 1 + \ell_{r,n}$ (i.e. $h = \frac{2}{\mathrm{Var}(\hat{b}_s/b)}$).

Englehardt and Bain (1977a) proposed, as a new simple unbiased estimator for complete samples to replace $\hat{b}_s$,

$$\hat{b}_{EB} = \frac{\left(-\sum_{i=1}^{s} y_{(i)} + \frac{s}{n-s}\sum_{i=s+1}^{n} y_{(i)}\right)}{nk_n}$$

where $s = [qn]$, $0 < q < 1$ is chosen to minimize the asymptotic variance of $\hat{b}_{EB}$. This results in

$$s = [.84n]$$

and

$$k_n = E\left[-\sum_{i=1}^{s} w_{(i)} + \frac{s}{n-s}\sum_{i=s+1}^{n} w_{(i)}\right]/n \ , \quad w_i \sim EV(0,1)$$

$$= -\left[E\sum_{i=1}^{s} w_{(i)} + s\gamma\right]/(n-s)$$

and γ is Euler's constant. Then

$$n\,\mathrm{Var}\left(\frac{\hat{b}_{EB}}{b}\right) = \frac{V_s + \left(\frac{s}{n-s}\right)U_{s+1} - \frac{s\pi^2}{6}}{(n-s)k_n^2}$$

where $V_s = \mathrm{Var}\left(\sum_{i=1}^{s} w_{(i)}\right)$, $U_{s+1} = \mathrm{Var}\left(\sum_{i=s+1}^{n} w_{(i)}\right)$, $\mathrm{Var}\left(\sum_{i=1}^{n} w_{(i)}\right) = \frac{n\pi^2}{6}$.

To obtain an approximate BLIE (or MLE) we may use $b^*_{EB} = \dfrac{\hat{b}_{EB}}{1+\mathrm{Var}\left(\frac{\hat{b}_{EB}}{b}\right)}$.

Three other simple estimators have been suggested. Menon (1963) used the transformation $Z = \ln\left(\frac{X}{\theta}\right)^{\eta}$ where $X \sim \mathrm{WEI}(\theta,\eta,0)$ and $\ln X \sim \mathrm{EV}_I(\xi,b)$. He suggested $\hat{b}_{MN} = \left(\frac{6s^2}{\pi^2}\right)^{1/2}$ and $\hat{\xi}_{MN} = \widehat{\ln\theta}$ $= \frac{\sum_{i=1}^{n} \ln X_i}{n} + \gamma\hat{b}_M$. Then $\hat{\eta}_{MN} = \frac{1}{\hat{b}_M} \overset{\cdot}{\sim} N\left(\eta, \frac{1.1\eta^2}{n}\right)$ and $\hat{\theta}_{MN} = e^{\hat{\xi}_M}$ $\sim N\left(\theta, \frac{1.2b^2\theta^2}{n}\right)$. Dubey (1967) suggested estimating η by using percentiles. Based upon two percentiles,

$$\hat{\eta}_D = \frac{\ln\{-\ln(1-p_1)\} - \ln\{-\ln(1-p_2)\}}{\ln y_{p_1} - \ln y_{p_2}}$$

where $p_1 = 17\%$ and $p_2 = 97\%$ minimize $\mathrm{Var}(\hat{\eta}_D)$ and

$$y_p = \begin{cases} x_{(np)} & \text{if } np \text{ is an integer} \\ x_{[np]+1} & \text{if } np \text{ is not an integer.} \end{cases}$$

Asymptotically $\hat{\eta}_D \overset{\cdot}{\sim} N\left(\eta, \frac{\eta^2(.91627479)}{n}\right)$.

Murthy and Swartz (1975) used an approach similar to Dubey but used the EV_I distribution and obtained an estimator of b based upon two order statistics

$$\hat{b}_{MS} = \{\ell n\, T_{(j)} - \ell_n\, T_{(\ell)}\}\, B(N, \ell, j)$$

where $B(N, \ell, j) = \frac{1}{2E(Y)}$ and $Y = \frac{\ell n\, T_{(j)} - \ell n\, T_{(\ell)}}{2b}$. This is MVUE and has relative efficiency w.r.t. the Cramer-Rao lower bound (CRLB) approaching 70%. The following Table 5.4.1 gives the optimum values for j and ℓ.

Table 5.4.1

Optimal Order Statistics for Murthy-Swartz Estimator of $b = \frac{1}{\eta}$

n	j	ℓ
$2 \leq n \leq 5$	n	1
$6 \leq n \leq 10$	n	2
$11 \leq n \leq 16$	n	3
$17 \leq n \leq 23$	n	4
$24 \leq n \leq 26$	n	5

5.4.2 Estimators suggested for use.

Under the exchangeable model with a shape change, we assume n-1

observations are governed by $f(x;\theta,\eta) = \frac{\eta x^{\eta-1}}{\theta^{\eta}} \exp\{-(\frac{x}{\theta})^{\eta}\}$, $x > 0$, $\eta, \theta > 0$ and one observation is governed by $f(x;\theta,k^*\eta)$, $0 < k^* \leq 1$. If $0 < k^* < 1$, we have exactly one spurious observation. We are interested in estimating η (or in the EV_I case, $b = \frac{1}{\eta}$).

From Table 5.4.1, for $2 \leq n \leq 26$, the optimal form of the Murthy-Swartz estimator uses the largest order statistic. Dubey's estimator involves one or both of the smallest and largest order statistics for $2 \leq n \leq 33$. Also $\hat{b}_s$, $2 \leq n \leq 15$, $\hat{b}_B$ and $\hat{b}_{EB}$ all involve these two highly suspect values. Menon's estimator $\hat{b}_{MN} = \sqrt{\frac{6s^2_{\ln x}}{\pi^2}}$ has

$$MSE(\hat{b}_{MN}) = Var(\hat{b}_{MN}) + \{Bias(\hat{b}_{MN})\}^2$$

$$= \frac{1.1\ b^2}{n} + b^2 O(\frac{1}{n^2}) + \{bO(\frac{1}{n})\}^2$$

under the homogeneous model. However, with the above exchangeable model,

$$E_{het}(\hat{b}_{MN}) = \frac{\sqrt{6}}{\pi}\{\sqrt{Var_{\ln x}} + bO(\frac{1}{n})\} \text{ [see Cramer (1946), 27.7.1]}$$

$$= \frac{\sqrt{6}}{\pi}\{\sqrt{\frac{\pi^2}{6} b^2 (\frac{n-1}{n}) + \frac{\pi^2}{6}(\frac{b}{k^*})^2 \frac{1}{n}} + bO(\frac{1}{n})\}$$

$$= b\sqrt{\frac{n-(1-\frac{1}{k^{*2}})}{n}} + bO(\frac{1}{n})$$

and as $k^* \to 0$, bias $\to \infty$.

Also $MSE_{het}(\hat{b}_{MN}) = Var_{het}(\hat{b}_{MN}) + \{Bias(\hat{b}_{MN})\}^2$ and

$$Var_{het}(\hat{b}_{MN}) = \frac{6}{\pi^2}\{\frac{\mu_4-\mu_2^2}{4n\mu_2} + O(\frac{1}{n^2})\} \quad \text{[see Cramer (1946), 27.7.2]}$$

$$= \frac{6}{\pi^2}\left[\frac{3b^2}{2\pi^2}\{\frac{\pi^4}{15}(\frac{n-1+\frac{1}{k^{*4}}}{n}) + \frac{4(2.404)\gamma(n-1+\frac{1}{k^{*3}})(n-1+\frac{1}{k^*})}{n^2}\right.$$

$$+ \frac{\pi^2\gamma^2(n-1+\frac{1}{k^{*2}})(n-1+\frac{1}{k^*})^2}{n^3} - \frac{3\gamma^4(n-1+\frac{1}{k^*})^4}{n^4} - \frac{\pi^4(n-1+\frac{1}{k^*})^2}{36n^2}\}$$

$$\left. + b^2O(\frac{1}{n^2})\right]$$

(see Appendix V).

Taking $\lim_{k^* \to 0} MSE_{het}(\hat{b}_{MN})$, we find $MSE(\hat{b}_{MN}) \to \infty$ as $k^* \to 0$. Thus this estimation technique seems poor if an outlier is present. Also $s^2_{\ell n\, x}$ would be strongly affected by the presence of an outlier occurring at $X_{(1)}$ or at $X_{(n)}$ and these are precisely the values where $u(r;n,k^*)$ is largest as $k^* \to 0$. We might obtain a more robust estimator by using trimming, winsorization or semi-winsorization. Thus we might consider

i) $\hat{b}_{RA} = \left(\frac{6\hat{\sigma}_A^2}{\pi^2}\right)^{1/2}$ and $\hat{\xi}_{RA} = \hat{\mu}_A + \gamma\hat{b}_{RA}$

ii) $\hat{b}_{RW} = \left(\frac{6\hat{\sigma}_W^2}{\pi^2}\right)^{1/2}$ and $\hat{\xi}_{RW} = \hat{\mu}_W + \gamma\hat{b}_{RW}$

iii) $\hat{b}_{RS} = \left(\frac{6\hat{\sigma}_S^2}{\pi^2}\right)^{1/2}$ and $\hat{\xi}_{RS} = \hat{\mu}_S + \gamma\hat{b}_{RS}$

where $Y_i = \ell n\, X_i$, $b = \frac{1}{\eta}$, $\xi = \ell n\, \theta$ and $\hat{\sigma}_A^2$, $\hat{\sigma}_W^2$, $\hat{\sigma}_S^2$, $\hat{\mu}_A$, $\hat{\mu}_W$ and $\hat{\mu}_S$ are as defined in Appendix I and γ is Euler's constant.

Table 5.4.2 Computation of means and variances and winsorized means and variances for 25 samples of size five based on $EV_I(0,1/\eta)$ with one spurious observation present (see Appendix I)

η = 5.00000 κ^* = 0.10000

Weibull (V)

0.102121E+01	0.861288E+00	0.125484E+01	0.812620E+00	0.222729E+01	
0.803152E+00	0.937162E+00	0.905811E+00	0.580656E+00	0.370609E+01	
0.962867E+00	0.100282E+01	0.123995E+01	0.841884E+00	0.423470E+00	
0.937319E+00	0.105347E+01	0.757929E+00	0.760034E+00	0.423634E-01	
0.927337E+00	0.877620E+00	0.762281E+00	0.123282E+01	0.127538E+01	
0.661348E+00	0.858771E+00	0.100320E+01	0.107187E+01	0.841635E-01	
0.820470E+00	0.106628E+01	0.120829E+01	0.989250E+00	0.493110E-02	
0.105657E+01	0.897128E+00	0.897488E+00	0.848710E+00	0.387857E+00	
0.138243E+01	0.725363E+00	0.926866E+00	0.698868E+00	0.359345E+00	
0.613861E+00	0.107227E+01	0.764260E+00	0.938452E+00	0.387784E+00	
0.937215E+00	0.895044E+00	0.549735E+00	0.310820E+00	0.134010E+01	
0.105084E+01	0.915291E+00	0.121398E+01	0.110028E+01	0.890449E-02	
0.624547E+00	0.931725E+00	0.121240E+01	0.110820E+01	0.252715E+01	
0.985510E+00	0.750880E+00	0.122693E+01	0.101768E+01	0.577123E+00	
0.809748E+00	0.860864E+00	0.703416E+00	0.113378E+01	0.902794E+00	
0.849408E+00	0.891488E+00	0.115158E+01	0.138763E+01	0.116568E+00	
0.752757E+00	0.869805E+00	0.107318E+01	0.115550E+01	0.111080E+01	
0.468859E+00	0.841906E+00	0.706769E+00	0.840323E+00	0.222881E-01	
0.116367E+01	0.727470E+00	0.124521E+01	0.984764E+00	0.340384E+00	
0.125350E+01	0.113521E+01	0.398613E+00	0.874528E+00	0.916387E+00	
0.738252E+00	0.982366E+00	0.894012E+00	0.104101E+01	0.485553E+01	
0.104629E+01	0.105262E+01	0.984382E+00	0.785419E+00	0.344385E-01	
0.104512E+01	0.118195E+01	0.101539E+01	0.100000E+01	0.281081E-01	
0.764350E+00	0.923114E+00	0.864547E+00	0.861185E+00	0.301828E-02	
0.902853E+00	0.810288E+00	0.773216E+00	0.904088E+00	0.261505E-01	

Extreme Value (W)

0.208857E-01	-0.188888E-01	0.227088E+00	-0.207481E+00	0.800785E+00	
0.219212E+00	-0.649100E-01	-0.989251E-01	-0.526522E+00	0.130988E+01	
-0.377159E-01	0.261685E-02	0.215074E+00	-0.584728E-01	-0.858273E+00	
0.647313E-01	0.520921E-01	-0.277166E+00	-0.287636E+00	-0.316147E+01	
-0.754381E-01	-0.130542E+00	-0.271440E+00	0.209382E+00	0.243242E+00	
-0.413474E+00	-0.154584E+00	0.318303E-02	0.884987E-01	-0.247489E+01	
-0.187078E+00	0.641838E-01	0.189202E+00	-0.108085E-01	-0.531218E+01	
0.550264E-01	-0.108555E+00	-0.108156E+00	-0.164038E+00	-0.100008E+01	
0.323842E+00	-0.321097E+00	-0.768860E-01	-0.385742E+00	-0.102347E+01	
-0.488019E+00	0.697750E-01	-0.268847E+00	-0.888587E-01	-0.100023E+01	
-0.648429E-01	-0.110882E+00	-0.598318E+00	-0.116854E+01	0.292745E+00	
0.496879E-01	-0.885127E-01	0.193914E+00	-0.855844E-01	-0.513204E+01	
-0.470729E+00	-0.707178E-01	0.182598E+00	-0.102741E+00	-0.827084E+00	
-0.145959E-01	-0.288510E+00	0.204514E+00	-0.175057E-01	-0.548889E+00	
-0.945874E-01	-0.150051E+00	-0.351806E+00	0.125558E+00	-0.102261E+00	
-0.598914E+00	-0.854804E-02	0.141132E+00	-0.313007E+00	-0.214930E+01	
-0.284013E+00	-0.306602E-01	0.706285E-01	-0.144831E+00	-0.105167E+00	
-0.757452E+00	-0.598498E-01	-0.347052E+00	-0.173889E+00	-0.380368E+01	
-0.151577E+00	-0.318183E+00	0.220108E+00	-0.571368E+00	-0.107768E+01	
0.225939E+00	0.126819E+00	-0.917260E+00	-0.133615E+00	-0.873161E-01	
-0.302117E+00	-0.177688E-01	-0.112037E+00	0.401938E-01	-0.160050E+01	
0.433347E-01	0.605533E-01	-0.187307E-01	-0.267331E+00	-0.338655E+01	
0.441355E-01	0.150088E+00	0.152692E-01	0.324247E-05	-0.357170E+01	
-0.268730E+00	-0.800028E-01	-0.145434E+00	-0.413722E+00	-0.580307E+01	
-0.102185E+00	-0.210364E+00	-0.257197E+00	-0.100829E+00	-0.364389E+01	

Sample	$\bar{w}$	s^2_w	$\bar{w}_{(2,1)}$	$s^2_{(2,1)w}$	$\bar{w}_{(5,1)}$	$s^2_{(5,1)w}$	$\bar{w}_{(4,5)}$	$s^2_{(4,5)w}$	$\bar{w}_{(1,5)}$	$s^2_{(1,5)w}$
1	0.16850	0.15035	0.20222	0.12236	0.38818	0.78814	0.04576	0.03585	-0.03718	0.03269
2	0.08008	0.50581	0.14154	0.43149	0.44738	0.82335	-0.19480	0.03839	-0.28722	0.05101
3	-0.14783	0.17008	0.01205	0.01353	0.06704	0.01876	-0.19033	0.14056	-0.36270	0.20599
4	-0.76778	1.84136	-0.17302	0.02488	-0.10507	0.02849	-0.77115	1.78737	-1.39049	2.82154
5	-0.00496	0.04882	0.02322	0.03502	0.09788	0.03423	-0.01173	0.04585	-0.10780	0.03893
6	-0.58407	1.14017	-0.18177	0.05137	-0.08517	0.04206	-0.80733	1.11805	-1.10287	1.59063
7	-1.05350	5.68732	-0.03064	0.02841	0.04678	0.02601	-1.07850	5.61277	-2.15378	8.32210
8	-0.26516	0.17580	-0.08785	0.00808	-0.05414	0.01045	-0.28778	0.15470	-0.47817	0.22823
9	-0.28867	0.24253	-0.16913	0.08212	-0.02721	0.11598	-0.37842	0.18031	-0.58614	0.18758
10	-0.35060	0.17637	-0.24815	0.06247	-0.13659	0.05780	-0.37768	0.15158	-0.58460	0.18046
11	-0.32997	0.32032	-0.21592	0.14626	-0.03771	0.13468	-0.40148	0.23456	-0.82222	0.29241
12	-0.97622	5.40723	-0.03243	0.01490	0.08891	0.01378	-0.89885	5.35172	-2.04147	7.86428
13	-0.13620	0.28025	-0.21620	0.17086	-0.41576	0.22684	-0.01070	0.07769	-0.14337	0.09825
14	-0.12576	0.08888	-0.07312	0.04495	-0.02509	0.04078	-0.18318	0.06298	-0.27660	0.07610
15	-0.11483	0.02884	-0.07428	0.01315	-0.01916	0.01780	-0.19868	0.01215	-0.21010	0.01718
16	-0.46052	1.00694	-0.15045	0.18056	-0.03184	0.14238	-0.49480	0.84837	-0.85298	1.26919
17	0.00113	0.02963	-0.05180	0.00635	-0.08684	0.00527	-0.00674	0.02712	-0.08458	0.03664
18	-1.02840	2.47696	-0.41916	0.10583	-0.27963	0.08516	-1.05122	2.42430	-1.77716	3.46716
19	-0.31911	0.28784	-0.21764	0.14707	-0.05955	0.13245	-0.33281	0.27030	-0.57866	0.27480
20	-0.15709	0.20265	-0.00036	0.02762	-0.07155	0.02851	-0.17691	0.18563	-0.38573	0.24509
21	0.24176	0.59379	-0.27877	0.54932	-0.62226	0.80038	-0.07031	0.02066	-0.13877	0.02519
22	-0.70955	2.22674	-0.08930	0.02721	-0.02372	0.01953	-0.71298	2.22021	-1.39537	3.25816
23	-0.67244	2.63022	-0.04180	0.00384	-0.07182	0.00534	-0.69263	2.58888	-1.41640	3.86992
24	-1.36218	5.23478	-0.26432	0.02319	-0.19758	0.02056	-1.35526	5.19435	-2.48680	8.17369
25	-0.86288	2.42150	-0.18555	0.00625	-0.15429	0.00554	-0.86317	2.42098	-1.57150	3.58214

We could seek M.L.E's of θ and η under the exchangeable model where

$$L(\underline{x};k^*,\eta,\theta) = \frac{k^*\eta^n}{n\theta^{\eta(n+k^*-1)}} e^{-\sum_{i=1}^{n}(\frac{x_i}{\theta})^{\eta}}$$

$$\prod_{i=1}^{n} x_i^{\eta-1}\{\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}\}$$

and

$$K(\underline{x};k^*,\eta,\theta) = \ln L(\underline{x};k^*,\eta,\theta)$$

$$= \ln k^* + n \ln \eta - \ln n - \eta(n+k^*-1)\ln \theta - \sum_{i=1}^{n} (x_i/\theta)^{\eta}$$

$$+ \sum_{i=1}^{n} (\eta-1)\ln x_i + \ln\{\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}\}.$$

Then

$$\frac{\partial K}{\partial \theta} = \frac{-\eta(n+k^*-1)}{\theta} + \sum_{i=1}^{n} \frac{\eta x_i^{\eta}}{\theta^{\eta+1}}$$

(continued)

$$+\frac{\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}} \{k^*\eta \frac{x_i^{k^*\eta}}{\theta^{k^*\eta+1}} - \frac{\eta x_i^{\eta}}{\theta^{\eta+1}}\}}{\sum_{i=1}^{n} x_i^{\eta(k^*+1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}}$$

$$\frac{\partial K}{\partial \eta} = \frac{n}{\eta} - (n+k^*-1)\ln\theta - \sum_{i=1}^{n} (\frac{x_i}{\theta})^{\eta} \ln(\frac{x_i}{\theta}) + \sum_{i=1}^{n} \ln x_i$$

$$+ [\sum_{i=1}^{n} x_i^{\eta(k^*-1)} (\ln x_i)(k^*-1) e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}$$

$$+ x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}} \{-k^*(\frac{x_i}{\theta})^{k^*\eta} \ln(\frac{x}{\theta}) + (\frac{x_i}{\theta})^{\eta} \ln(\frac{x_i}{\theta})\}]$$

$$[\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}]^{-1}$$

$$\frac{\partial K}{\partial k^*} = \frac{1}{k^*} - \eta \ln\theta + [\sum_{i=1}^{n} x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}]^{-1}$$

$$\left[\sum_{i=1}^{n} [x_i^{\eta(k^*-1)} (\ln x_i)\eta e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}}\right.$$

$$\left. + x_i^{\eta(k^*-1)} e^{-(x_i/\theta)^{k^*\eta}+(x_i/\theta)^{\eta}} \{-\eta (x_i/\theta)^{k^*\eta} \ln(\frac{x_i}{\theta})\}]\right]$$

The above equations appear mathematically intractable.

Bibliography

Anscombe, F.J. (1960). 'Rejection of outliers'. Technometrics, 2, 123-147.

Apostol, T.M. (1964). Mathematical Analysis. Addison-Wesley, London.

Ash, R. (1972). Real Analysis and Probability. Academic Press, New York.

Bain, L.J. (1973). 'Results for one or more independent censored samples from the Weibull or Extreme-Value distribution'. Technometrics, 15, 279-288.

Bain, L.J. (1978). Statistical Analysis of Reliability and Life-testing Models. Marcel Dekker, New York.

Bain, L.J., and Engelhardt, M.E. (1975). 'A two moment Chi-square approximation for the statistic log $(\bar{x}/\tilde{x})$'. J. Amer. Statist. Ass., 70, 948-954.

Barnett, V., and Lewis, T. (1978). Outliers in Statistical Data. John Wiley, New York.

Basu, A.P. (1965). 'On some tests of hypotheses relating to the exponential distribution when some outliers are present' J. Amer. Statist. Ass., 60, 548-559.

Box, G.E.P., and Tiao, G.C. (1968). 'A Bayesian approach to some outlier problems'. Biometrika, 55, 119-129.

Chauvenet, W. (1863). 'Method of least squares'. Appendix to Manual of Spherical and Practical Astronomy, Vol. 2, Lippincott, Philadelphia, pp. 469-566; tables 593-599. Reprinted (1960) 5th edition. Dover, New York.

Chikkagoudar, M.S., and Kunchur, S.H. (1980). 'Estimation of the mean of an exponential distribution in the presence of an outlier'. Can. J. Statist., 8, 59-63.

Choi, S.C., and Wette, R. (1969). 'Maximum likelihood estimation of the parameters of the Gamma distribution and their bias'. Technometrics, 11, 683-690.

Cohen, A. Clifford (1965). 'Maximum likelihood estimation in the Weibull distribution based on complete and on censored samples'. Technometrics, 7, 579-588.

Copson, E.T. (1967). Asymptotic Expansions. Cambridge University Press.

Cramer, H. (1946). Mathematical Methods of Statistics. Princeton University Press.

De Brüijn, N.G. (1961). Asymptotic Methods in Analysis, 2nd Edition. North Holland, Amsterdam.

Dempster, A.P., and Rosner, B. (1971). 'Detection of outliers'. In Gupta and Yackel (1971).

Dingle, R.B. (1973). Asymptotic Expansions. Academic Press, New York.

Dubey, S.D. (1967). 'Some percentile estimators for Weibull parameters'. Technometrics, 9, 119-129.

Englehardt, M. (1975). 'On simple estimation of the parameters of the Weibull or Extreme-Value distribution'. Technometrics, 17, 369-374.

Englehardt, M., and Bain, L.J. (1973). 'Some complete and censored sampling results for the Weibull or Extreme-Value distributions'. Technometrics, 15, 541-549.

Englehardt, M., and Bain, L.J. (1977a). 'Simplified statistical procedures for the Weibull or Extreme-Value distribution'. Technometrics, 19, 323-331.

Englehardt, M., and Bain, L.J. (1977b). 'Uniformly most powerful unbiased tests on the scale-parameter of a Gamma distribution with a nuisance shape parameter'. Technometrics, 19, 77-82.

Englehardt, M., and Bain, L.J. (1978). 'Construction of optimal unbiased inference procedures for the parameters of the gamma distribution'. Technometrics, 20, 485-489.

Ferguson, T.S. (1961). 'On the rejection of outliers'. Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Vol. 1, 253-287.

Fieller, N.R.J. (1976). 'Discordancy test for single upper outlier in Gamma when $\beta = 1/2$'. Ph.D. thesis, Department of Mathematical Statistics, University of Hull.

Goldthwaithe, L. (1961). 'Failure rate study for the Lognormal lifetime model'. Proceedings of the Seventh National Symposium on Reliability and Quality Control, 208-213.

Gradshteyn, I.S., and Ryzhik, I.M. (1980). Table of Integrals, Series and Products, Corrected and Enlarged Edition. Academic Press, New York.

Green, R.F. (1974). 'A note on outlier-prone families of distributions'. Ann. Math. Statist., 2, 1293-1295.

Green, R.F. (1976), 'Outlier-prone and outlier-resistant distributions'. J. Amer. Statist. Ass., 71, 502-505.

Grubbs, F.E. (1950). 'Sample Criteria for testing outlying observations'. Ann. Math. Statist., 21, 27-58.

Grubbs, F.E. (1969). 'Procedures for detecting outlying observations in samples'. Technometrics, 11, 1-21.

Gupta, R.P. (Ed.) (1975). Applied Statistics. North Holland, Amsterdam.

Gupta, S.S., and Yackel, J. (Eds.)(1971). Statistical Decision Theory and Related Topics. Academic Press, New York.

Guttman, I. (1973a). 'Premium and protection of several procedures for dealing with outliers when sample sizes are moderate to large'. Technometrics, 15, 385-404.

Guttman, I. (1973b). 'Care and handling of univariate and multivariate outliers in detecting spuriosity - a Bayesian approach'. Technometrics, 15, 723-738.

Guttman, I., and Smith, D.E. (1969). 'Investigation of rules for dealing with outliers in small samples from the normal distribution. I: Estimation of the mean'. Technometrics, 11, 527-550.

Guttman, I., and Smith, D.E. (1971). 'Investigation of rules for dealing with outliers in small samples from the normal distribution. II: Estimation of the variance'. Technometrics, 13, 101-111.

Harter, H.L., and Moore, A.H. (1965). 'Maximum likelihood estimation of the parameters of Gamma and Weibull populations from complete and from censored samples'. Technometrics, 7, 639-643.

Hassanein, K.M. (1969). 'Estimation of the parameters of the Extreme-Value distribution by use of two or three order statistics'. Biometrika, 56, 429-436.

Hassanein, K.M. (1972). 'Simultaneous estimation of the parameters of the Extreme-Value distribution by sample quantiles'. Technometrics, 14, 63-70.

Hawkins, D.M. (1980). Identification of Outliers. Chapman and Hall, England.

Howard, B.T., and Dodson, G.A. (1961). 'High stress aging to failure of semiconductor devices'. Proceedings of the Seventh National Symposium on Reliability and Quality Control.

Jahnke, Emde and Losch (1960). Tables of Higher Functions, 6th edition. McGraw-Hill, New York.

Joshi, P.C. (1972a). 'Some slippage tests of mean for a single outlier in linear regression'. Biometrika, 59, 109-120.

Joshi, P.C. (1972b). 'Efficient estimation of a mean of an exponential distribution when an outlier is present'. Technometrics, 14, 137–143.

Kale, B.K. (1969). 'Estimation of expected life in the presence of an outlier observation. A Bayesian approach'. Contributed paper, 37th session of ISI meeting London, England. 150-152.

Kale, B.K. (1974a). 'Trimmed means and the method of MLE when spurious observations are present'. Technical Reprot. No. 53, Department of Statistics, University of Manitoba, Canada. In Gupta (1975), 177-186.

Kale, B.K. (1974b). 'Detectin of outliers'. Technical Report No. 63, Department of Statistics, University of Manitoba, Canada. Paper presented at the International Symposium on Recent Trends of Research in Statistics, held at I.S.I., Calcutta, December, 1974. Sankhya, Ser. B, 38, 356-363.

Kale, B.K. (1974c). 'Detection of outliers - a semi-Bayesian approach' (preliminary report)(abstract). Inst. Math. Statist. Bull., 3, 153.

Kale, B.K. (1975a). 'A note on outlier - resistant families and mixtures of distributions'. Technical Report No. 66, Department of Statistics, University of Manitoba, Canada.

Kale, B.K. (1975b). 'On outlier proneness of some families of distributions'. Technical Report No. 68, Department of Statistics, University of Manitoba, Canada.

(The above two papers appear in combined form in Jour. Ind. Stat. Assn.. (1977,15).)

Kale, B.K., and Sinha, S.K. (1971). 'Estimation of expected life in the presence of an outlier observation'. Technometrics, 13, 755–759.

Kao, J.H.K. (1975). 'Graphical estimation of mixed Weibull parameters in life testing electron tubes'. Technometrics, 1, 389-407.

Kao, J.H.K. (1965). 'Statistical models in mechanical reliability'. Proceedings of the Eleventh National Symposium on Reliability and Quality Control. 240-247.

Kimber, A.C. (1979). 'Tests for a single outlier in a Gamma sample with unknown shape and scale'. Applied Statistics, 28, 243-250.

Kudo, A. (1956). 'On the testing of outlying observations'. Sankhya, 18, 163-166.

Laurent, A.G. (1963). 'Conditional distribution of order statistics and distribution of the reduced i^{th} order statistic of the exponential model'. Ann. Math. Statist., 34, 652-657.

Lehmann, E.L. (1959). Testing Statistical Hypotheses. John Wiley, New York.

Lieblein, J., and Zelen, M. (1956). 'Statistical investigation of the fatigue life of deep grove ball bearings'. Research paper 2719. Journal of Res. of Nat. Bureau of Standards, 57, 273-316.

Lilliefors, H.W. (1971). 'Reducing the bias of estimates of parameters for the Erlang and Gamma distributions'. In Mann, Schafer and Singpurwalla (1974).

Linhart, H. (1965). 'Approximate confidence limits for the coefficient of variation of Gamma distributions'. Biometrics, 31, 733-737.

McCool, J.I. (1970). 'Inference on Weibull percentiles and shape parameter from maximum likelihood estimators'. IEEE Transactions in Reliability R-19, 2-9.

Mann, N.R. (1967), 'Tables for obtaining the best linear invariant estimators of the parameters of the Weibull distribution'. Technometrics, 9, 629-645.

Mann, N.R. (1968). 'Point and interval estimation procedures for the two-parameter Weibull and Extreme-Value distributions'. Technometrics, 10, 231-256.

Mann, N.R., and Fertig, K.W. (1973). 'Tables for obtaining Weibull confidence bounds and tolerance bounds based on best linear invariant estimates of parameters of the Extreme-Value distribution'. Technometrics, 15, 87-101.

Mann, N.R., and Fertig, K.W. (1975). 'Simplified efficient point and interval estimators for Weibull parameters'. Technometrics, 17, 361-368.

Mann, N.R., and Fertig, K.W. (1977). 'Efficient unbiased quantile estimators for moderate-size complete samples from Extreme-Value and Weibull distributions; confidence bounds and tolerance and prediction intervals'. Technometrics, 19, 87-93.

Mann, N.R., Schafer, R.E., and Singpurwalla, N.D. (1974). Methods for Statistical Analysis of Reliablity and Life Data. John Wiley, New York.

Menon, M.V. (1963). 'Estimation of the shape and scale parameters of the Weibull distribution'. Technometrics, 5, 175-182.

Mount, K.S., and Kale, B.K. (1973). 'On selecting a spurious observation'. Can. Math. Bull., 16, 75-78.

Murthy, V.K., and Swartz, G.B. (1975). 'Estimation of Weibull parameters from two order statistics'. J. Roy. Statist. Soc., B, 37, 96-102.

Nelson, L.S. (1977). 'Using 10^{th} and 90^{th} percentiles and the mode to characterize Lognormal distributions'. Ann. Math. Statist., 31, 26-30.

Neyman, J. (1979). 'Outlier-proneness and resistance - developments in probability and mathematical statistics generated by studies in meteorology and weather'. Comm. St. A, 8, 1097-1110..

Neyman, J., and Scott, E.L. (1971). 'Outlier-proneness of phenomena and of related distributions'. In Rustagi (1971).

Olver, F.W.J. (1974). Introduction to Asymptotics and Special Functions. Academic Press, New York.

Patel, J.K., Kapadia, C.H., and Owen, D.B. (1976). Handbook of Statistical Distributions. Marcel Dekker, New York.

Patnaik, P.B. (1949), 'The non-central χ^2 and F distributions and their applications'. Biometrika, 36, 202-232.

Paulson, E. (1952). 'An optimum solution to the k sample slippage problem for the normal distribution'. Ann. Math. Statist., 37, 78-87.

Pearson, K., and Pearson, M.V. (1931). 'On the mean character and variance of a ranked individual and on the mean and the variance of the intervals between ranked individuals'. Biometrika, 23, 364-397.

Pearson, E.S., and Chandrasekar, C. (1936). 'The efficiency of statistical tools and a criterion for the rejection of outlying observations'. Biometrika, 28, 308-320.

Peck, D.S. (1961). 'Semiconductor reliability predictions from life distribution data'. Proceedings of the AGET Conference on Reliability of Semiconductor Devices.

Rauhut, B.O. (1982). 'Estimation of the mean of the Exponential distribution with an outlying observation'. Comm. Statist., Theory and Methods, 11, 1439-1452.

Rustagi, J. (Ed.)(1971). Optimizing Methods in Statistics. Academic Press, New York.

Singpurwalla, N.D., and Keubler, A.A. (1966). 'A quantitative evaluation of drill life'. Amer Soc. of Mech. Egineers Pub. No. 66 - WA/PROD-11.

Sinha, S.K. (1973a). 'Some distributions relevant in life testing when an outlier may be present'. Technical Report No. 42, Department of Statistics, University of Manitoba, Canada.

Sinha, S.K. (1973b). 'Estimation of the parameters of a two-parameter exponential distribution when an outlier may be present'. Utilitas Mathemtica, 3, 75-82. Correction (1974), Utilitas Mathematica, 4, 333-334.

Sinha, S.K. (1973c). 'Life-testing and reliability estimation for non-homogeneous data - a Bayesian approach'. Comm. Statist., 2, 235–243.

Sinha, S.K. (1973d). 'Distributions of order statistics and estimation of mean life when an outlier may be present'. Can. J. of Statist., 1, 119-121.

Sinha, S.K. (1972). 'Reliability estimation in life - testing in the presence of an outlier observation'. Op Res., 20, 888-894.

Thom, H.C.S. (1968). 'Approximate convolution of the Gamma and mixed Gamma distributions'. Monthly Weather Review, 96(12), 883-886.

Thoman, D.R., Bain, L.J., and Antle, C.E. (1969). 'Inferences on the parameters of the Weibull distribution'. Technometrics, 11, 445–460.

Thoman, D.R., Bain, L.J. and Antle, C.E. (1970). 'Maximum likelihood estimation, exact confidence intervals for reliability and tolerance limits in the Weibull distribution'. Technometrics, 12, 363-371.

Veale, J.R. (1975). "Improved estimation of expected life when one identified spurious observation may be present'. J. Amer. Statist. Assn., 70, 398-401.

Veale, J.R., and Huntsberger, D.V. (1969). 'Estimation of a mean when one observation may be spurious'. Technometrics, 11, 331-339.

Veale, J.R., and Kale, B.K. (1972). 'Tests of hypothesis for expected life in the presence of a spurious observation'. Utilitas Mathematics, 2, 8-23.

Weibull, W. (1939). 'A statistical theory of the strength of materials'. Ing. Vetenskaps. Akad. Handl., No. 151, 1-45.

Weibull, W. (1951). 'A statistical distribution function of wide applicability'. J. Appl. Mechanics, 18, 292-297.

White, John S. (1969), 'The moments of log-Weibull order statistics'. Technometrics, 11, 373-386.

Wilks, S.S. (1963). 'Multivariate statistical outliers'. Sankhya, Ser. A, 25, 407-425.

Appendix I: The A-, W-, and S-Rules and Premium/Protection.

We adopt the following notation:

$$z_i = y_i - \bar{y} \qquad \bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} \qquad s^2 = \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n-1}$$

$$\bar{y}_{(1)} = \frac{\sum_{i=2}^{n} y_{(i)}}{n-1} \qquad \bar{y}_{(n)} = \frac{\sum_{i=1}^{n-1} y_{(i)}}{n-1} \qquad \bar{y}_{(j,k)} = \frac{\left(\sum_{i=1}^{n} y_{(i)} + y_{(j)} - y_{(k)}\right)}{n}$$

$$s^2_{(1)} = \frac{\sum_{i=2}^{n} (y_{(i)} - \bar{y}_{(1)})^2}{n-2} \qquad s^2_{(n)} = \frac{\sum_{i=1}^{n-1} (y_{(i)} - \bar{y}_{(n)})^2}{n-2}$$

$$s^2_{(j,k)} = \frac{1}{(n-1)} \left\{ \sum_{i=1}^{n} (y_{(i)} - \bar{y}_{(j,k)})^2 + (y_{(j)} - \bar{y}_{(j,k)})^2 - (y_{(k)} - \bar{y}_{(j,k)})^2 \right\}.$$

Anscombe's Rule (A-Rule):

$$\mu_A = \begin{cases} \bar{y} & \text{if } |z_{(n)}| < Cs \text{ and } |z_{(1)}| < Cs \\ \bar{y}_{(1)} & \text{if } |z_{(1)}| \geq Cs \text{ and } |z_{(1)}| > |z_{(n)}| \\ \bar{y}_{(n)} & \text{if } |z_{(n)}| \geq Cs \text{ and } |z_{(n)}| > |z_{(1)}| \end{cases}.$$

$$\sigma^2_A = \begin{cases} Ds^2 & \text{if } z^2_{(1)} < Ks^2 \text{ and } z^2_{(n)} < Ks^2 \\ Ds^2_{(1)} & \text{if } z^2_{(1)} \geq Ks^2 \text{ and } z^2_{(1)} > z^2_{(n)} \\ Ds^2_{(n)} & \text{if } z^2_{(n)} \geq Ks^2 \text{ and } z^2_{(n)} \geq z^2_{(1)} \end{cases}.$$

Winsorization (W-Rule):

$$\mu_W = \begin{cases} \bar{y} & \text{if } |z_{(1)}| < Cs \text{ and } |z_{(n)}| < Cs \\ \bar{y}_{(2,1)} & \text{if } |z_{(1)}| \geq Cs \text{ and } |z_{(1)}| > |z_{(n)}| \\ \bar{y}_{(n-1,n)} & \text{if } |z_{(n)}| \geq Cs \text{ and } |z_{(n)}| > |z_{(1)}| \end{cases} .$$

$$\sigma_W^2 = \begin{cases} Ds^2 & \text{if } z_{(1)}^2 < Ks^2 \text{ and } z_{(n)}^2 < Ks^2 \\ D \max[s_{(2,1)}^2, s_{(n,1)}^2] & \text{if } z_{(1)}^2 \geq Ks^2 \text{ and } z_{(1)}^2 > z_{(n)}^2 \\ D \max[s_{(1,n)}^2, s_{(n-1,n)}^2] & \text{if } z_{(n)}^2 \geq Ks^2 \text{ and } z_{(n)}^2 > z_{(1)}^2 \end{cases} .$$

Semi-Winsorization (S-Rule):

$$\mu_S = \begin{cases} \bar{y} & \text{if } |z_{(1)}| < Cs \text{ and } |z_{(n)}| < Cs \\ \dfrac{(n-1)\bar{y}_{(1)}+(\bar{y}-Cs)}{n} & \text{if } |z_{(1)}| \geq Cs \text{ and } |z_{(1)}| > |z_{(n)}| \\ \dfrac{(n-1)\bar{y}_{(n)}+(\bar{y}+Cs)}{n} & \text{if } |z_{(n)}| \geq Cs \text{ and } |z_{(n)}| > |z_{(1)}| \end{cases} .$$

$$\sigma_S^2 = \begin{cases} Ds^2 & \text{if } z_{(1)}^2 < Ks^2 \text{ and } z_{(n)}^2 < Ks^2 \\ \frac{D}{n-1}\{(n-2)s_{(1)}^2 + Ks^2\} & \text{if } z_{(1)}^2 \geq Ks^2 \text{ and } z_{(1)}^2 > z_{(n)}^2 \\ \frac{D}{n-1}\{(n-2)s_{(n)}^2 + Ks^2\} & \text{if } z_{(n)}^2 \geq Ks^2 \text{ and } z_{(n)}^2 > z_{(1)}^2 \end{cases} .$$

If we adopt the premium-protection approach suggested by Anscombe (1960) we define

$$\text{Premium} = \frac{\text{MSE(new estimator)}-\text{MSE(old estimator)}}{\text{MSE(old estimator)}}$$

assuming homogeneous data

and

$$\text{Protection} = \frac{\text{MSE(old estimator)}-\text{MSE(new estimator)}}{\text{MSE(old estimator)}}$$

assuming spurious values(s) are present.

Appendix II: Asymptotic approximation for $\Gamma(\nu+1)$

Consider $\Gamma(\nu+1) = \int_0^\infty e^{-u}u^\nu\,du$. Setting $u = \nu t$, $du = \nu dt$ and $\Gamma(\nu+1) = \int_0^\infty e^{-\nu t}(\nu t)^\nu \nu dt = \nu^{\nu+1}\int_0^\infty e^{\nu(-t+\ell n\ t)}dt$. This last integral is of the form $\int_0^\infty \vartheta(t)e^{\nu h(t)}dt$ where ν is a large positive constant and $\vartheta(t)$ and $h(t)$ are real and continuous in $[0,\infty)$. Now $h(t) = -t + \ell n\ t$ which has a single maximum at $t = 1$ with $h'(1) = 0$, $h''(1) = -1$. Applying the Laplace approximation to the two intervals $0 \le t \le 1$ and $1 \le t < \infty$ we obtain the following:

$$\int_0^\infty e^{\nu(-t+\ell n\ t)}dt = \int_0^1 e^{\nu(-t+\ell n\ t)}dt + \int_1^\infty e^{\nu(-t+\ell n\ t)}dt\ .$$

Since $h(t)$ has a maximum at $t = 1$, the main contributions to the two integrals on the right occur in the neighborhood of $t = 1$. We may rewrite $\int_0^1 e^{\nu(-t+\ell n\ t)}dt = \int_0^{1-\varepsilon} e^{\nu(-t+\ell nt)}dt + \int_{1-\varepsilon}^1 e^{\nu(-t+\ell n\ t)}dt$ and $\int_1^\infty e^{\nu(-t+\ell n\ t)}dt = \int_1^{1+\delta} e^{\nu(-t+\ell n\ t)}dt + \int_{1+\delta}^\infty e^{\nu(-t+\ell n\ t)}dt$. Setting $x^2 = h(1) - h(t)$, $2xdx = -h'(t)dt$ and expanding $h(t)$ in a Taylor's Series, $h(t) = h(1) + h'(1)(t-1) + h''(\xi)\frac{(t-1)^2}{2!}$, $1 < \xi < 1+\delta$, we obtain $x^2 = h(1) - h(t) = -h''(\xi)\frac{(t-1)^2}{2}$ and

$$h'(t) = h''(\xi)(t-1) \quad .$$

Now

$$\frac{2x}{h'(t)} = \frac{2\sqrt{-h''(\xi)\,\frac{(t-1)^2}{2}}}{h''(\xi)(t-1)}$$

$$= \frac{-1}{\sqrt{-\frac{1}{2}\,h''(\xi)}}$$

$$= \frac{-1}{\sqrt{-\frac{1}{2}\,h''(1)}} \quad .$$

Thus

$$\int_{t=1}^{1+\delta} e^{\nu(-t+\ell n\ t)}dt = \int_{x=0}^{\tau} e^{\nu(-1-x^2)}\left\{\frac{-2x}{h'(t)}\right\} dx$$

$$= e^{-\nu}\int_{\tau}^{0} e^{-\nu x^2}\frac{-1}{\sqrt{-\frac{1}{2}\,h''(1)}}\,dx$$

$$= e^{-\nu}\sqrt{2}\int_{0}^{\tau} e^{-\nu x^2}\,dx \quad .$$

But $\int_0^{\tau} e^{-\nu x^2}dx \doteq \int_0^{\infty} e^{-\nu x^2}\,dx$ since the major contribution to this

latter integral occurs in the neighborhood of $x = 0$. Thus

$$\int_1^{1+\delta} e^{\nu(-t+\ln t)}dt \doteq e^{-\nu}\sqrt{2}\int_0^{\infty} e^{-\nu x^2}dx$$

$$= e^{-\nu}\sqrt{2}\left(\frac{1}{2}\sqrt{\frac{\pi}{\nu}}\right)$$

$$= e^{-\nu}\sqrt{\frac{\pi}{2\nu}} \quad .$$

Similarly we can show $\int_{1-\varepsilon}^{1} e^{\nu(-t+\ln t)}dt \doteq e^{-\nu}\sqrt{\frac{\pi}{2\nu}}$ and thus

$$\Gamma(\nu+1) \doteq \nu^{\nu+1}\left\{\int_{1-\varepsilon}^{1} e^{\nu(-t+\ln t)}\,dt + \int_1^{1+\delta} e^{\nu(-t+\ln t)}dt\right\}$$

$$= \nu^{\nu+1}\, 2e^{-\nu}\sqrt{\frac{\pi}{2\nu}}$$

$$= e^{-\nu}\nu^{\nu}\sqrt{2\pi\nu} \quad .$$

Appendix III: Evaluation of $\int_{-\infty}^{\infty} \phi(x)\Phi(vx)dx$.

Theorem: $\int_{-\infty}^{\infty} \phi(x)\Phi(vx)dx = 1/2$.

Proof: Let $f(v) = \int_{-\infty}^{\infty} \phi(x)\Phi(vx)dx$.

Then $f'(v) = \int_{-\infty}^{\infty} \phi(x)\phi(vx)xdx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-1/2(x^2+v^2x^2)}xdx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1+v^2}} \int_{-\infty}^{\infty} \frac{\sqrt{1+v^2}}{\sqrt{2\pi}} e^{-\frac{x^2}{2}(1+v^2)}xdx$$

and setting $u = x\sqrt{1+v^2}$ we obtain

$$f'(v) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1+v^2}} \int_{-\infty}^{\infty} z \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$$

$$= 0.$$

Then $f(v) = c$ for all v.

In particular, if $v = 1$,

$$f(1) = c = \int_{-\infty}^{\infty} \phi(x)\Phi(x)dx = \frac{\Phi(x)^2}{2}\Big|_{-\infty}^{\infty} = 1/2.$$

Thus $f(v) = 1/2$ for all v.

i.e. $\int_{-\infty}^{\infty} \phi(x)\Phi(vx)dx = 1/2$.

Appendix IV

Tables of $u(r;n,k^*) = P(X_{(r)}$ is spurious in a sample of $n|k^*)$ where the exchangeable model of Weibulls with at most one outlier (involving shape change) is assumed.

K* = .01050
N = 50

n												
1	1.00000											
2	.62988	.37011										
3	.62721	.00535	.36743									
4	.62585	.00470	.00334	.36632								
5	.62464	.00444	.00273	.00263	.36555							
6	.62368	.00431	.00248	.00205	.00226	.36521						
7	.62297	.00422	.00237	.00182	.00171	.00202	.36487					
8	.62238	.00417	.00228	.00171	.00148	.00151	.00188	.36461				
9	.62185	.00412	.00223	.00164	.00137	.00127	.00138	.00173	.36439			
10	.62141	.00408	.00218	.00159	.00131	.00120	.00110	.00128	.00183	.36421		
11	.62100	.00407	.00218	.00154	.00127	.00110	.00100	.00110	.00120	.00154	.36406	
12	.62063	.00405	.00213	.00151	.00123	.00110	.00095	.00091	.00099	.00110	.00146	.36392
13	.62030	.00403	.00211	.00148	.00120	.00100	.00092	.00084	.00084	.00095	.00110	.00140 .36381
14	.61999	.00402	.00210	.00146	.00120	.00100	.00091	.00081	.00078	.00079	.00092	.00100 .00134 .36370
15	.61970 .36361	.00401	.00209	.00145	.00110	.00098	.00088	.00080	.00072	.00089	.00078	.00089 .00099 .00129
16	.61943 .00128	.00400	.00208	.00144	.00110	.00095	.00087	.00079	.00071	.00084	.00085	.00075 .00088 .00095 .36353
17	.61918 .00090	.00399	.00207	.00143	.00110	.00093	.00084	.00078	.00071	.00083	.00058	.00083 .00073 .00084 .00120 .36345
18	.61895 .00081	.00398	.00206	.00142	.00110	.00091	.00081	.00078	.00070	.00083	.00057	.00085 .00081 .00072 .00086 .00120 .36338
19	.61873 .00071	.00397	.00205	.00142	.00110	.00088	.00079	.00074	.00070	.00083	.00055	.00082 .00083 .00081 .00078 .00082 .00120 .36332
20	.61852 .00060	.00396	.00204	.00141	.00110	.00088	.00077	.00071	.00068	.00083	.00057	.00081 .00048 .00051 .00070 .00075 .00078 .00110 .36328
21	.61832 .00051	.00396	.00204	.00140	.00110	.00087	.00075	.00068	.00068	.00083	.00058	.00081 .00048 .00048 .00060 .00068 .00072 .00075 .00110 .36320
22	.61814 .00044	.00395	.00203	.00140	.00110	.00087	.00074	.00067	.00064	.00082	.00058	.00082 .00048 .00043 .00050 .00060 .00067 .00069 .00072 .00110 .36315
23	.61798 .00040	.00395	.00202	.00139	.00110	.00087	.00073	.00065	.00062	.00081	.00058	.00083 .00047 .00042 .00043 .00050 .00060 .00068 .00069 .00110 .36310
24	.61778 .00038	.00394	.00202	.00139	.00110	.00087	.00072	.00064	.00060	.00058	.00057	.00083 .00048 .00042 .00038 .00038 .00042 .00051 .00060 .00064 .00083 .00067 .00110 .36308
25	.61782 .00039	.00394	.00201	.00138	.00110	.00087	.00072	.00062	.00058	.00057	.00058	.00084 .00049 .00043 .00039 .00038 .00037 .00042 .00051 .00060 .00062 .00061 .00065 .00110 .36301
26	.61746 .00039	.00394	.00201	.00138	.00110	.00087	.00072	.00062	.00058	.00055	.00054	.00083 .00049 .00044 .00038 .00038 .00034 .00037 .00043 .00051 .00058 .00060 .00058 .00083 .00100 .36297
27	.61731 .00040	.00394	.00201	.00137	.00110	.00087	.00073	.00061	.00055	.00053	.00053	.00082 .00050 .00045 .00038 .00033 .00033 .00038 .00043 .00052 .00054 .00054 .00058 .00081 .00100 .36293
28	.61717 .00041	.00393	.00200	.00138	.00110	.00087	.00073	.00061	.00054	.00051	.00051	.00081 .00050 .00048 .00038 .00033 .00031 .00032 .00036 .00044 .00052 .00056 .00055 .00053 .00080 .00100 .36288
29	.61703 .00042	.00393	.00200	.00138	.00110	.00087	.00073	.00062	.00053	.00049	.00049	.00049 .00049 .00047 .00037 .00033 .00031 .00030 .00032 .00037 .00044 .00052 .00055 .00053 .00051 .00058 .00100 .36286
30	.61689 .00043	.00393	.00200	.00138	.00100	.00087	.00073	.00062	.00053	.00048	.00047	.00047 .00048 .00047 .00038 .00034 .00031 .00029 .00028 .00031 .00037 .00045 .00052 .00054 .00051 .00049 .00057 .00100 .36282
31	.61678 .00044	.00393	.00200	.00138	.00100	.00087	.00074	.00062	.00053	.00047	.00045	.00048 .00047 .00048 .00040 .00035 .00031 .00029 .00028 .00028 .00031 .00038 .00048 .00052 .00052 .00048 .00048 .00058 .00100 .36279
32	.61663 .00044	.00392	.00199	.00138	.00100	.00088	.00074	.00063	.00053	.00047	.00044	.00044 .00045 .00045 .00041 .00036 .00032 .00029 .00027 .00027 .00028 .00032 .00038 .00048 .00051 .00050 .00047 .00048 .00055 .00100 .36275
33	.61651 .00044	.00392	.00199	.00134	.00100	.00088	.00074	.00063	.00054	.00048	.00043	.00042 .00043 .00044 .00041 .00037 .00033 .00028 .00027 .00028 .00028 .00027 .00032 .00039 .00047 .00050 .00048 .00048 .00055 .00100 .36272
34	.61639 .00043	.00392	.00199	.00134	.00100	.00088	.00074	.00064	.00054	.00048	.00042	.00040 .00041 .00043 .00042 .00038 .00034 .00030 .00027 .00028 .00028 .00028 .00027 .00033 .00040 .00047 .00048 .00047 .00043 .00043 .00054 .00100 .36269
35	.61628 .00042	.00392	.00199	.00134	.00100	.00088	.00074	.00064	.00054	.00047	.00041	.00038 .00039 .00041 .00042 .00039 .00035 .00031 .00028 .00024 .00024 .00028 .00028 .00033 .00041 .00047 .00048 .00048 .00041 .00042 .00053 .00100 .36268

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36   .81616 .00392 .00199 .00134 .00100 .00084 .00073 .00064 .00055 .00047 .00041 .00038 .00038 .00039
     .00041 .00041 .00040 .00038 .00032 .00028 .00026 .00024 .00023 .00023 .00024 .00028 .00034 .00042
     .00047 .00047 .00043 .00039 .00041 .00053 .00100 .38263

37   .81606 .00391 .00199 .00134 .00100 .00084 .00073 .00064 .00055 .00047 .00041 .00037 .00038 .00037
     .00039 .00040 .00040 .00037 .00033 .00029 .00028 .00024 .00023 .00022 .00022 .00024 .00028 .00035
     .00042 .00046 .00045 .00041 .00038 .00041 .00052 .00100 .38260

38   .81595 .00391 .00199 .00134 .00100 .00083 .00073 .00064 .00055 .00048 .00041 .00037 .00035 .00036
     .00038 .00039 .00040 .00038 .00035 .00030 .00027 .00025 .00023 .00022 .00022 .00022 .00024 .00029
     .00036 .00043 .00046 .00044 .00039 .00037 .00040 .00052 .00100 .38258

39   .81585 .00391 .00198 .00134 .00100 .00083 .00072 .00064 .00055 .00048 .00041 .00037 .00034 .00034
     .00036 .00036 .00039 .00038 .00035 .00032 .00028 .00025 .00023 .00022 .00021 .00021 .00022 .00024
     .00030 .00037 .00043 .00045 .00042 .00037 .00035 .00039 .00051 .00100 .38255

40   .81575 .00391 .00198 .00133 .00100 .00082 .00072 .00064 .00057 .00048 .00042 .00037 .00034 .00033
     .00034 .00038 .00038 .00038 .00038 .00033 .00029 .00026 .00023 .00022 .00021 .00021 .00020 .00021
     .00025 .00031 .00038 .00043 .00044 .00040 .00036 .00034 .00038 .00051 .00100 .38252

41   .81565 .00391 .00198 .00133 .00100 .00082 .00071 .00064 .00057 .00048 .00042 .00037 .00033 .00032
     .00033 .00035 .00037 .00037 .00037 .00034 .00030 .00026 .00024 .00022 .00021 .00020 .00020 .00020
     .00021 .00025 .00032 .00038 .00043 .00043 .00039 .00034 .00033 .00038 .00051 .00100 .38250

42   .81555 .00391 .00198 .00133 .00100 .00081 .00071 .00064 .00057 .00050 .00043 .00037 .00033 .00031
     .00031 .00033 .00036 .00037 .00037 .00035 .00031 .00027 .00024 .00022 .00021 .00020 .00020 .00019
     .00019 .00022 .00028 .00033 .00039 .00043 .00042 .00037 .00033 .00033 .00038 .00051 .00100 .38248

43   .81548 .00391 .00198 .00133 .00100 .00081 .00070 .00063 .00057 .00050 .00043 .00037 .00033 .00031
     .00030 .00031 .00033 .00035 .00036 .00035 .00032 .00028 .00025 .00022 .00021 .00020 .00020 .00019
     .00019 .00019 .00022 .00027 .00034 .00040 .00042 .00040 .00036 .00031 .00032 .00037 .00050 .00100
     .38245

44   .81537 .00390 .00198 .00133 .00100 .00081 .00070 .00063 .00057 .00051 .00044 .00038 .00033 .00030
     .00029 .00030 .00032 .00034 .00035 .00035 .00033 .00030 .00028 .00023 .00021 .00020 .00020 .00019
     .00018 .00018 .00019 .00022 .00028 .00034 .00040 .00042 .00039 .00034 .00030 .00031 .00037 .00050
     .00100 .38243

45   .81526 .00390 .00198 .00133 .00100 .00080 .00069 .00063 .00057 .00051 .00045 .00038 .00033 .00030
     .00029 .00029 .00030 .00032 .00034 .00035 .00034 .00031 .00027 .00024 .00021 .00020 .00019 .00019
     .00018 .00018 .00018 .00019 .00023 .00028 .00035 .00040 .00041 .00037 .00032 .00028 .00031 .00037
     .00050 .00100 .38240

46   .81519 .00390 .00198 .00133 .00100 .00080 .00069 .00062 .00057 .00051 .00045 .00039 .00034 .00030
     .00028 .00028 .00029 .00031 .00033 .00034 .00034 .00032 .00028 .00025 .00022 .00020 .00018 .00019
     .00018 .00018 .00017 .00017 .00019 .00024 .00030 .00036 .00040 .00040 .00036 .00031 .00028 .00030
     .00038 .00050 .00100 .38236

47   .81511 .00390 .00198 .00133 .00100 .00080 .00068 .00062 .00057 .00052 .00046 .00039 .00034 .00030
     .00028 .00027 .00028 .00028 .00032 .00033 .00034 .00033 .00030 .00026 .00023 .00020 .00019 .00018
     .00018 .00018 .00017 .00017 .00017 .00020 .00024 .00031 .00037 .00040 .00038 .00034 .00028 .00027
     .00030 .00036 .00049 .00100 .38236

48   .81503 .00390 .00198 .00133 .00100 .00080 .00068 .00061 .00057 .00052 .00046 .00040 .00034 .00030
     .00027 .00028 .00028 .00028 .00030 .00032 .00033 .00033 .00031 .00027 .00024 .00021 .00018 .00018
     .00018 .00018 .00017 .00017 .00018 .00017 .00020 .00026 .00032 .00037 .00040 .00038 .00033 .00028
     .00028 .00030 .00038 .00049 .00100 .38234

49   .81495 .00390 .00197 .00133 .00100 .00080 .00067 .00061 .00056 .00052 .00046 .00040 .00035 .00030
     .00027 .00026 .00026 .00027 .00028 .00031 .00033 .00033 .00031 .00028 .00025 .00022 .00018 .00018
     .00018 .00018 .00017 .00017 .00018 .00018 .00017 .00021 .00026 .00032 .00038 .00038 .00037 .00031
     .00027 .00026 .00030 .00036 .00049 .00100 .38232

50   .81487 .00390 .00197 .00133 .00100 .00080 .00067 .00060 .00056 .00052 .00047 .00041 .00035 .00031
     .00027 .00026 .00025 .00025 .00027 .00030 .00032 .00033 .00032 .00030 .00028 .00023 .00020 .00018
     .00017 .00017 .00017 .00017 .00018 .00018 .00018 .00017 .00021 .00027 .00033 .00038 .00038 .00038
     .00030 .00026 .00025 .00029 .00035 .00048 .00100 .38230
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K* = .01100
N = 50

n														
1	1.00000													
2	.62979	.37021												
3	.62888	.00581	.36741											
4	.62534	.00492	.00349	.36625										
5	.62418	.00486	.00286	.00275	.36556									
6	.62327	.00481	.00281	.00215	.00236	.36509								
7	.62254	.00443	.00248	.00191	.00180	.00212	.36473							
8	.62191	.00436	.00240	.00179	.00158	.00158	.00194	.36448						
9	.62137	.00432	.00234	.00172	.00144	.00134	.00144	.00181	.36423					
10	.62090	.00428	.00229	.00166	.00137	.00120	.00120	.00133	.00170	.36404				
11	.62047	.00426	.00226	.00162	.00132	.00120	.00110	.00110	.00125	.00161	.36388			
12	.62008	.00424	.00223	.00158	.00128	.00110	.00100	.00098	.00100	.00120	.00154	.36374		
13	.61973	.00422	.00221	.00155	.00125	.00110	.00097	.00088	.00089	.00099	.00110	.00147	.36362	
14	.61941	.00421	.00220	.00153	.00120	.00110	.00095	.00085	.00080	.00084	.00095	.00110	.00141	.36351
15	.61911	.00420	.00219	.00152	.00120	.00100	.00092	.00084	.00078	.00074	.00081	.00092	.00100	.00137
	.36341													
16	.61883	.00419	.00218	.00151	.00120	.00095	.00090	.00082	.00074	.00068	.00070	.00078	.00089	.00097
	.00133	.36332												
17	.61857	.00418	.00216	.00150	.00120	.00097	.00087	.00081	.00074	.00067	.00063	.00087	.00077	.00086
	.00093	.00128	.36324											
18	.61832	.00417	.00216	.00148	.00120	.00095	.00085	.00079	.00073	.00065	.00060	.00089	.00085	.00075
	.00083	.00088	.00126	.36317										
19	.61809	.00416	.00215	.00148	.00110	.00094	.00082	.00077	.00072	.00064	.00059	.00065	.00086	.00084
	.00074	.00080	.00085	.00124	.36310									
20	.61787	.00415	.00214	.00148	.00110	.00093	.00080	.00074	.00071	.00064	.00060	.00064	.00051	.00055
	.00083	.00072	.00076	.00081	.00120	.36303								
21	.61767	.00415	.00213	.00147	.00110	.00092	.00079	.00072	.00069	.00065	.00060	.00064	.00049	.00049
	.00054	.00063	.00071	.00073	.00078	.00120	.36297							
22	.61747	.00414	.00213	.00146	.00110	.00092	.00078	.00070	.00066	.00064	.00060	.00055	.00049	.00046
	.00047	.00054	.00063	.00065	.00070	.00075	.00120	.36292						
23	.61728	.00414	.00212	.00146	.00110	.00092	.00077	.00068	.00064	.00063	.00060	.00055	.00049	.00046
	.00043	.00048	.00054	.00062	.00067	.00067	.00072	.00120	.36287					
24	.61710	.00413	.00211	.00145	.00110	.00092	.00077	.00067	.00062	.00061	.00059	.00055	.00050	.00046
	.00041	.00041	.00046	.00054	.00061	.00065	.00064	.00070	.00120	.36281				
25	.61693	.00413	.00211	.00144	.00110	.00092	.00077	.00066	.00060	.00058	.00058	.00055	.00051	.00046
	.00041	.00039	.00040	.00046	.00054	.00061	.00062	.00062	.00068	.00110	.36277			
26	.61677	.00413	.00211	.00144	.00110	.00092	.00077	.00066	.00059	.00056	.00056	.00055	.00052	.00047
	.00042	.00038	.00037	.00039	.00046	.00054	.00060	.00060	.00059	.00066	.00110	.36272		
27	.61661	.00412	.00210	.00143	.00110	.00092	.00077	.00065	.00058	.00054	.00054	.00054	.00052	.00048
	.00043	.00038	.00035	.00035	.00038	.00048	.00054	.00059	.00058	.00057	.00065	.00110	.36268	
28	.61645	.00412	.00210	.00143	.00110	.00091	.00077	.00065	.00057	.00053	.00052	.00052	.00051	.00048
	.00044	.00039	.00035	.00034	.00035	.00039	.00047	.00054	.00057	.00066	.00054	.00063	.00110	.36264
29	.61631	.00412	.00210	.00142	.00110	.00091	.00077	.00065	.00057	.00052	.00050	.00050	.00056	.00048
	.00045	.00040	.00036	.00033	.00032	.00034	.00039	.00047	.00054	.00065	.00053	.00062	.00062	.00110
	.36260													
30	.61617	.00411	.00209	.00142	.00110	.00091	.00077	.00065	.00057	.00051	.00048	.00048	.00049	.00048
	.00045	.00041	.00036	.00033	.00031	.00031	.00034	.00040	.00048	.00053	.00054	.00051	.00051	.00061
	.00110	.36256												
31	.61603	.00411	.00209	.00142	.00110	.00090	.00077	.00065	.00057	.00050	.00047	.00045	.00047	.00047
	.00045	.00042	.00037	.00033	.00031	.00029	.00030	.00034	.00041	.00048	.00052	.00052	.00049	.00049
	.00060	.00110	.36252											
32	.61590	.00411	.00208	.00141	.00110	.00090	.00077	.00065	.00057	.00050	.00045	.00045	.00048	.00046
	.00046	.00043	.00038	.00034	.00031	.00028	.00028	.00030	.00034	.00041	.00048	.00051	.00050	.00047
	.00048	.00060	.00110	.36248										
33	.61577	.00411	.00209	.00141	.00110	.00088	.00077	.00065	.00057	.00050	.00045	.00043	.00044	.00045
	.00045	.00043	.00040	.00035	.00031	.00029	.00027	.00028	.00030	.00035	.00042	.00048	.00050	.00048
	.00045	.00048	.00058	.00110	.36245									
34	.61564	.00411	.00209	.00141	.00110	.00088	.00077	.00065	.00058	.00050	.00044	.00042	.00042	.00043
	.00044	.00043	.00040	.00036	.00032	.00029	.00027	.00028	.00027	.00030	.00035	.00043	.00048	.00049
	.00048	.00043	.00045	.00054	.00110	.36241								
35	.61552	.00410	.00209	.00141	.00110	.00088	.00078	.00065	.00058	.00050	.00044	.00041	.00040	.00041
	.00043	.00043	.00041	.00037	.00033	.00030	.00027	.00028	.00025	.00027	.00030	.00036	.00043	.00048
	.00048	.00044	.00042	.00044	.00054	.00110	.36238							

36	.61540	.00410	.00208	.00141	.00110	.00088	.00076	.00067	.00063	.00061	.00044	.00040	.00039	.00040	
	.00041	.00042	.00041	.00038	.00034	.00031	.00028	.00026	.00025	.00025	.00026	.00031	.00037	.00044	
	.00047	.00046	.00042	.00040	.00043	.00057	.00110	.38235							
37	.61529	.00410	.00208	.00140	.00110	.00087	.00076	.00067	.00063	.00061	.00044	.00040	.00038	.00038	
	.00040	.00041	.00041	.00039	.00036	.00032	.00028	.00026	.00024	.00024	.00026	.00031	.00036		
	.00044	.00047	.00045	.00041	.00039	.00042	.00057	.00110	.38232						
38	.61514	.00410	.00208	.00140	.00110	.00087	.00075	.00067	.00059	.00052	.00045	.00039	.00037	.00037	
	.00038	.00040	.00040	.00039	.00036	.00033	.00029	.00026	.00024	.00023	.00023	.00024	.00027	.00032	
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6	.62006	.00615	.00356	.00294	.00322	.36407									
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8	.61820	.00595	.00326	.00244	.00214	.00214	.00285	.36321							
9	.61747	.00589	.00318	.00233	.00197	.00184	.00193	.00248	.36290						
10	.61682	.00584	.00313	.00228	.00186	.00168	.00165	.00178	.00234	.36264					
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18	.61331	.00568	.00294	.00203	.00158	.00132	.00110	.00100	.00097	.00091	.00087	.00087	.00090	.00096	.00100 .00123 .00183 .36140
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	.00051	.00049	.00050	.00057	.00074	.00100	.00145	.38025						
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	.00049	.00048	.00048	.00048	.00047	.00045	.00042	.00038	.00037	.00037	.00039	.00043	.00047	.00050
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	.00048	.00048	.00045	.00046	.00048	.00045	.00043	.00041	.00038	.00035	.00035	.00038	.00039	.00043
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	.00043	.00056	.00074	.00091	.00129	.38986								
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	.00036	.00043	.00056	.00074	.00091	.00128	.38983							
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	.00049	.00046	.00042	.00038	.00036	.00034	.00034	.00035	.00036	.00037	.00037	.00036	.00034	.00032
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5	.61768	.00846	.00520	.00500	.36365										
6	.61804	.00820	.00475	.00382	.00429	.36280									
7	.61470	.00804	.00450	.00349	.00328	.00385	.36218								
8	.61357	.00783	.00435	.00328	.00285	.00285	.00354	.36166							
9	.61259	.00785	.00425	.00311	.00263	.00248	.00257	.00331	.36124						
10	.61173	.00779	.00417	.00301	.00248	.00224	.00219	.00237	.00313	.36090					
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13	.60961	.00767	.00403	.00283	.00226	.00194	.00176	.00166	.00165	.00173	.00198	.00276	.36011		
14	.60903	.00764	.00400	.00280	.00222	.00189	.00169	.00157	.00152	.00154	.00184	.00190	.00267	.35990	
15	.60848	.00762	.00397	.00277	.00218	.00184	.00164	.00151	.00144	.00141	.00145	.00158	.00183	.00260	.35972
16	.60797	.00760	.00395	.00274	.00215	.00181	.00159	.00148	.00137	.00133	.00133	.00137	.00149	.00177	.00253 .35955
17	.60750	.00758	.00393	.00272	.00213	.00178	.00156	.00142	.00132	.00126	.00124	.00125	.00131	.00143	.00171 .00247 .35935
18	.60706	.00755	.00391	.00270	.00211	.00176	.00153	.00138	.00128	.00120	.00120	.00120	.00120	.00125	.00138 .00166 .00242 .35925
19	.60664	.00755	.00390	.00269	.00209	.00174	.00151	.00135	.00125	.00120	.00110	.00110	.00110	.00110	.00120 .00134 .00182 .00237 .35912
20	.60624	.00754	.00388	.00267	.00207	.00172	.00149	.00133	.00120	.00110	.00110	.00110	.00100	.00110	.00110 .00120 .00130 .00158 .00233 .35900
21	.60585	.00753	.00387	.00266	.00206	.00170	.00147	.00131	.00120	.00110	.00110	.00100	.00099	.00099	.00100 .00100 .00110 .00126 .00155 .00229 .35888
22	.60551	.00752	.00386	.00265	.00204	.00169	.00146	.00129	.00120	.00110	.00100	.00099	.00095	.00094	.00095 .00097 .00100 .00110 .00123 .00151 .00225 .35877
23	.60516	.00751	.00385	.00264	.00203	.00168	.00144	.00128	.00120	.00110	.00100	.00098	.00093	.00091	.00090 .00081 .00084 .00088 .00110 .00120 .00145 .00222 .35867
24	.60484	.00750	.00384	.00263	.00202	.00166	.00143	.00127	.00110	.00110	.00099	.00094	.00091	.00088	.00086 .00086 .00088 .00091 .00085 .00100 .00120 .00146 .00218 .35852
25	.60453	.00749	.00384	.00262	.00201	.00165	.00142	.00125	.00110	.00100	.00097	.00092	.00088	.00085	.00083 .00082 .00083 .00085 .00088 .00092 .00100 .00120 .00143 .00215 .35849
26	.60423	.00748	.00383	.00261	.00201	.00165	.00141	.00124	.00110	.00100	.00095	.00090	.00087	.00084	.00081 .00079 .00079 .00081 .00083 .00085 .00089 .00098 .00110 .00141 .00213 .35840
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29	.60340	.00747	.00381	.00259	.00199	.00162	.00138	.00120	.00110	.00100	.00092	.00086	.00082	.00078	.00076 .00074 .00072 .00070 .00071 .00072 .00074 .00076 .00078 .00083 .00093 .00110 .00134 .00206 .35817
30	.60314	.00746	.00381	.00259	.00198	.00162	.00138	.00120	.00110	.00098	.00091	.00085	.00081	.00077	.00075 .00072 .00070 .00068 .00068 .00068 .00070 .00072 .00074 .00076 .00081 .00092 .00110 .00132 .00203 .35810
31	.60289	.00746	.00380	.00258	.00197	.00161	.00137	.00120	.00110	.00098	.00091	.00085	.00080	.00076	.00073 .00071 .00069 .00067 .00066 .00065 .00067 .00068 .00070 .00072 .00074 .00080 .00091 .00110 .00130 .00201 .35803
32	.60265	.00745	.00380	.00258	.00197	.00161	.00137	.00120	.00110	.00097	.00090	.00084	.00079	.00075	.00072 .00070 .00068 .00066 .00064 .00063 .00064 .00065 .00067 .00069 .00070 .00072 .00079 .00090 .00100 .00128 .00200 .35797
33	.60242	.00745	.00379	.00257	.00197	.00160	.00136	.00120	.00110	.00097	.00089	.00083	.00078	.00074	.00071 .00068 .00067 .00065 .00063 .00062 .00061 .00062 .00064 .00066 .00067 .00068 .00070 .00077 .00088 .00100 .00127 .00198 .35791
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	.00068	.00065	.00063	.00062	.00061	.00059	.00057	.00056	.00056	.00056	.00058	.00060	.00061	.00062
	.00063	.00067	.00075	.00086	.00098	.00122	.00183	.35774						
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39	.60116	.00743	.00377	.00255	.00194	.00158	.00134	.00120	.00100	.00084	.00088	.00080	.00075	.00071
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43	.60042	.00742	.00378	.00254	.00193	.00157	.00132	.00120	.00100	.00082	.00084	.00078	.00073	.00069
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	.00110	.00183	.35731											
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	.00085	.00110	.00182	.35727										
47	.59976	.00741	.00375	.00253	.00192	.00158	.00131	.00110	.00100	.00081	.00083	.00077	.00071	.00067
	.00063	.00061	.00058	.00055	.00053	.00050	.00048	.00048	.00047	.00047	.00047	.00048	.00044	.00043
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	.00063	.00060	.00058	.00055	.00052	.00050	.00048	.00047	.00048	.00048	.00046	.00048	.00044	.00043
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	.00069	.00075	.00084	.00110	.00180	.35720								
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	.00063	.00060	.00057	.00055	.00052	.00050	.00048	.00046	.00045	.00045	.00045	.00045	.00044	.00043
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	.00081	.00088	.00074	.00083	.00110	.00179	.35716							
50	.59929	.00740	.00375	.00253	.00192	.00155	.00131	.00110	.00100	.00081	.00083	.00078	.00071	.00066
	.00062	.00059	.00057	.00055	.00052	.00050	.00047	.00046	.00045	.00044	.00045	.00045	.00044	.00043
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6	.59207	.02038	.01185	.00979	.01070	.35519									
7	.58874	.01998	.01120	.00872	.00814	.00981	.35358								
8	.58593	.01967	.01080	.00814	.00712	.00713	.00883	.35233							
9	.58360	.01946	.01060	.00777	.00656	.00614	.00643	.00825	.35130						
10	.58136	.01929	.01040	.00751	.00621	.00580	.00548	.00592	.00781	.35043					
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	.00632	.34706													
17	.57091	.01872	.00976	.00678	.00530	.00444	.00390	.00354	.00330	.00318	.00310	.00313	.00327	.00358	
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18	.56981	.01867	.00972	.00673	.00525	.00438	.00383	.00345	.00320	.00303	.00294	.00292	.00297	.00313	
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19	.56877	.01863	.00968	.00669	.00520	.00433	.00377	.00338	.00312	.00293	.00282	.00276	.00276	.00284	
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22	.56599	.01853	.00959	.00658	.00510	.00421	.00363	.00323	.00294	.00273	.00258	.00247	.00240	.00237	
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27	.56213	.01841	.00948	.00648	.00498	.00406	.00348	.00308	.00277	.00254	.00236	.00222	.00212	.00203	
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	.00193	.00188	.00185	.00184	.00185	.00188	.00193	.00202	.00215	.00237	.00272	.00341	.00520	.34381	
29	.56079	.01837	.00944	.00644	.00494	.00405	.00345	.00303	.00273	.00249	.00231	.00216	.00205	.00196	
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32	.55895	.01832	.00940	.00640	.00490	.00400	.00341	.00298	.00267	.00243	.00224	.00209	.00197	.00188	
	.00180	.00173	.00168	.00165	.00162	.00161	.00161	.00162	.00164	.00168	.00175	.00185	.00199	.00221	
	.00256	.00324	.00489	.34312											
33	.55838	.01831	.00939	.00638	.00489	.00399	.00339	.00297	.00266	.00241	.00223	.00207	.00195	.00185	
	.00177	.00171	.00165	.00161	.00158	.00156	.00156	.00156	.00157	.00160	.00165	.00171	.00181	.00195	
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	.00175	.00168	.00163	.00158	.00155	.00153	.00151	.00151	.00151	.00153	.00156	.00161	.00168	.00178	
	.00192	.00214	.00250	.00318	.00490	.34282									
35	.55729	.01828	.00937	.00637	.00487	.00397	.00337	.00294	.00263	.00239	.00219	.00204	.00192	.00181	
	.00173	.00165	.00160	.00155	.00152	.00149	.00148	.00147	.00147	.00147	.00148	.00153	.00158	.00165	
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22	.44317	.04970	.02705	.01899	.01487	.01238	.01070	.00959	.00876	.00815	.00770	.00739	.00719	.00709
	.00711	.00726	.00758	.00818	.00918	.01120	.01862	.30008						
23	.44092	.04950	.02693	.01888	.01476	.01228	.01060	.00947	.00862	.00800	.00753	.00718	.00695	.00681
	.00677	.00682	.00700	.00734	.00793	.00897	.01100	.01837	.29933					
24	.43877	.04932	.02681	.01878	.01467	.01218	.01050	.00936	.00850	.00785	.00738	.00702	.00676	.00658
	.00649	.00648	.00657	.00677	.00712	.00773	.00877	.01080	.01813	.29863				
25	.43673	.04914	.02670	.01868	.01458	.01209	.01040	.00926	.00840	.00775	.00725	.00687	.00659	.00639
	.00627	.00621	.00623	.00634	.00656	.00693	.00754	.00855	.01080	.01591	.28797			
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	.00606	.00599	.00598	.00601	.00613	.00637	.00675	.00737	.00842	.01040	.01570	.28734		
27	.43289	.04881	.02650	.01852	.01442	.01183	.01030	.00909	.00821	.00755	.00704	.00664	.00633	.00609
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	.00971	.01484	.29482											
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	.00783	.00959	.01470	.29415										
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36	.41882	.04782	.02580	.01786	.01391	.01140	.00978	.00859	.00770	.00701	.00648	.00601	.00565	.00535
	.00510	.00490	.00472	.00459	.00447	.00439	.00432	.00428	.00427	.00427	.00431	.00437	.00448	.00462
	.00484	.00514	.00556	.00620	.00724	.00918	.01418	.28243						
37	.41780	.04751	.02574	.01781	.01387	.01140	.00974	.00855	.00786	.00697	.00642	.00597	.00561	.00530
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	.00454	.00475	.00505	.00548	.00612	.00716	.00908	.01407	.28204					
38	.41821	.04741	.02567	.01786	.01383	.01140	.00971	.00852	.00782	.00693	.00638	.00593	.00558	.00528
	.00500	.00478	.00481	.00446	.00434	.00424	.00416	.00410	.00406	.00404	.00404	.00406	.00411	.00418
	.00430	.00446	.00488	.00498	.00541	.00605	.00708	.00900	.01398	.28188				
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	.00496	.00474	.00486	.00441	.00428	.00417	.00409	.00402	.00397	.00394	.00393	.00394	.00397	.00402
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	.00469	.00446	.00426	.00408	.00394	.00381	.00370	.00360	.00352	.00345	.00339	.00334	.00330	.00327
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16	.40941	.06364	.03607	.02597	.02078	.01770	.01572	.01444	.01364	.01324	.01322	.01388	.01482	.01743	.02472 .28563
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20	.39508	.06177	.03487	.02492	.01975	.01661	.01454	.01310	.01207	.01130	.01080	.01050	.01040	.01050	.01090 .01180 .01288 .01554 .02265 .28018
21	.39201	.06137	.03462	.02471	.01956	.01642	.01433	.01287	.01182	.01110	.01050	.01020	.00995	.00992	.01010 .01040 .01120 .01252 .01518 .02224 .27905
22	.38910	.06100	.03439	.02452	.01938	.01624	.01415	.01268	.01160	.01080	.01020	.00983	.00957	.00945	.00948 .00987 .01010 .01080 .01219 .01485 .02187 .27801
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24	.38372	.06032	.03398	.02419	.01907	.01594	.01384	.01235	.01130	.01040	.00980	.00934	.00900	.00877	.00865 .00884 .00875 .00901 .00948 .01030 .01180 .01428 .02120 .27611
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26	.37884	.05970	.03381	.02388	.01881	.01568	.01358	.01208	.01100	.01010	.00947	.00898	.00858	.00829	.00809 .00788 .00794 .00800 .00817 .00847 .00895 .00979 .01120 .01379 .02063 .27441
27	.37655	.05942	.03344	.02376	.01869	.01557	.01347	.01196	.01080	.00988	.00933	.00881	.00841	.00810	.00788 .00773 .00785 .00785 .00773 .00792 .00824 .00876 .00958 .01080 .01387 .02037 .27383
28	.37436	.05914	.03328	.02364	.01858	.01548	.01336	.01188	.01070	.00987	.00920	.00868	.00826	.00797	.00781 .00761 .00740 .00736 .00739 .00749 .00770 .00803 .00855 .00938 .01080 .01336 .02012 .27289
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30	.37024	.05863	.03298	.02340	.01837	.01527	.01317	.01170	.01080	.00967	.00899	.00844	.00801	.00765	.00738 .00717 .00701 .00691 .00686 .00687 .00693 .00707 .00731 .00766 .00820 .00903 .01040 .01299 .01988 .27150
31	.36828	.05838	.03284	.02330	.01828	.01518	.01308	.01160	.01080	.00958	.00889	.00834	.00790	.00754	.00725 .00703 .00686 .00673 .00665 .00663 .00665 .00674 .00688 .00713 .00750 .00804 .00887 .01020 .01282 .01947 .27085
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33	.36480	.05792	.03258	.02310	.01811	.01502	.01294	.01140	.01030	.00942	.00873	.00817	.00771	.00734	.00703 .00679 .00659 .00644 .00632 .00626 .00622 .00623 .00628 .00639 .00656 .00682 .00720 .00775 .00889 .00995 .01251 .01908 .26983
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36	.26191	.06725	.04105	.03030	.02435	.02055	.01793	.01599	.01451	.01334	.01240	.01160	.01100	.01050
	.01000	.00958	.00936	.00910	.00890	.00874	.00862	.00855	.00852	.00853	.00860	.00872	.00891	.00919
	.00959	.01010	.01090	.01212	.01402	.01754	.02648	.22104						
37	.26005	.06864	.04081	.03012	.02421	.02043	.01781	.01588	.01440	.01323	.01230	.01150	.01090	.01040
	.00991	.00954	.00922	.00895	.00875	.00857	.00844	.00835	.00830	.00828	.00831	.00839	.00853	.00873
	.00901	.00941	.00997	.01080	.01185	.01385	.01735	.02622	.22031					
38	.25825	.06844	.04058	.02994	.02406	.02031	.01789	.01577	.01430	.01313	.01220	.01140	.01080	.01030
	.00980	.00942	.00910	.00883	.00861	.00842	.00828	.00817	.00810	.00808	.00807	.00811	.00820	.00834
	.00855	.00885	.00926	.00981	.01060	.01178	.01368	.01717	.02600	.21961				
39	.25652	.06806	.04035	.02976	.02383	.02018	.01768	.01567	.01420	.01304	.01210	.01130	.01070	.01020
	.00970	.00931	.00899	.00871	.00848	.00829	.00813	.00801	.00792	.00787	.00785	.00786	.00792	.00802
	.00817	.00839	.00869	.00910	.00966	.01050	.01180	.01382	.01700	.02578	.21893			
40	.25483	.06768	.04013	.02961	.02379	.02007	.01748	.01557	.01410	.01294	.01201	.01120	.01080	.01010
	.00960	.00921	.00888	.00860	.00838	.00818	.00800	.00787	.00777	.00770	.00766	.00765	.00768	.00774
	.00785	.00801	.00824	.00854	.00895	.00952	.01030	.01150	.01337	.01643	.02558	.21827		
41	.25320	.06732	.03992	.02946	.02366	.01998	.01738	.01547	.01401	.01286	.01193	.01120	.01050	.00997
	.00951	.00912	.00879	.00850	.00825	.00806	.00788	.00773	.00762	.00754	.00749	.00748	.00747	.00750
	.00758	.00789	.00786	.00809	.00840	.00882	.00939	.01020	.01140	.01323	.01667	.02538	.21764	
42	.25161	.06697	.03971	.02930	.02354	.01985	.01728	.01538	.01383	.01277	.01184	.01110	.01040	.00989
	.00943	.00903	.00869	.00840	.00815	.00794	.00776	.00761	.00749	.00740	.00733	.00729	.00728	.00729
	.00734	.00742	.00765	.00772	.00795	.00827	.00869	.00926	.01010	.01120	.01309	.01652	.02518	.21702
43	.25007	.06663	.03951	.02916	.02342	.01975	.01719	.01530	.01384	.01289	.01176	.01100	.01040	.00991
	.00934	.00895	.00861	.00831	.00805	.00784	.00765	.00750	.00737	.00727	.00719	.00714	.00711	.00711
	.00713	.00719	.00728	.00741	.00759	.00782	.00814	.00858	.00914	.00993	.01110	.01286	.01637	.02500
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44	.24858	.06628	.03931	.02901	.02330	.01965	.01710	.01521	.01376	.01282	.01170	.01090	.01030	.00973
	.00927	.00887	.00853	.00823	.00797	.00775	.00756	.00740	.00726	.00715	.00707	.00700	.00696	.00694
	.00695	.00698	.00704	.00714	.00728	.00748	.00770	.00802	.00845	.00902	.00982	.01100	.01284	.01623
	.02482	.21585												
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	.00920	.00880	.00845	.00815	.00789	.00767	.00747	.00730	.00715	.00705	.00695	.00688	.00682	.00679
	.00678	.00680	.00684	.00691	.00701	.00715	.00734	.00759	.00791	.00833	.00891	.00970	.01080	.01271
	.01610	.02485	.21529											
46	.24571	.06565	.03894	.02874	.02304	.01945	.01692	.01505	.01361	.01247	.01150	.01080	.01010	.00958
	.00913	.00873	.00838	.00808	.00781	.00758	.00738	.00722	.00707	.00695	.00684	.00678	.00670	.00666
	.00664	.00664	.00668	.00671	.00678	.00689	.00703	.00723	.00747	.00780	.00823	.00880	.00960	.01080
	.01260	.01597	.02448	.21475										
47	.24433	.06535	.03876	.02861	.02297	.01938	.01684	.01498	.01354	.01240	.01150	.01070	.01010	.00953
	.00906	.00865	.00831	.00801	.00774	.00751	.00731	.00713	.00698	.00685	.00674	.00665	.00658	.00654
	.00650	.00649	.00650	.00653	.00658	.00666	.00677	.00692	.00712	.00737	.00770	.00813	.00870	.00949
	.01070	.01249	.01584	.02432	.21422									
48	.24299	.06505	.03858	.02846	.02287	.01927	.01678	.01490	.01347	.01234	.01140	.01080	.01000	.00946
	.00900	.00859	.00824	.00794	.00767	.00744	.00723	.00705	.00690	.00677	.00665	.00655	.00648	.00642
	.00638	.00636	.00635	.00637	.00640	.00645	.00655	.00668	.00682	.00701	.00727	.00760	.00803	.00880
	.00960	.01080	.01238	.01572	.02417	.21371								
49	.24169	.06475	.03841	.02835	.02277	.01918	.01668	.01483	.01340	.01227	.01140	.01080	.00995	.00940
	.00894	.00853	.00818	.00788	.00761	.00737	.00716	.00698	.00682	.00668	.00657	.00647	.00639	.00632
	.00627	.00624	.00622	.00622	.00625	.00629	.00635	.00644	.00658	.00672	.00692	.00717	.00750	.00793
	.00851	.00930	.01050	.01227	.01581	.02402	.21320							
50	.24041	.06247	.03824	.02823	.02267	.01910	.01661	.01476	.01334	.01221	.01130	.01050	.00988	.00935
	.00888	.00848	.00812	.00782	.00755	.00731	.00710	.00691	.00675	.00661	.00648	.00639	.00630	.00623
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	.04095	.22749													
17	.28075	.08416	.05342	.04054	.03345	.02902	.02807	.02405	.02270	.02188	.02154	.02173	.02259	.02463	
	.02869	.03987	.22500												
18	.27587	.08285	.05258	.03986	.03282	.02840	.02642	.02335	.02181	.02095	.02042	.02030	.02084	.02181	
	.02362	.02780	.03889	.22271											
19	.27134	.08163	.05178	.03922	.03225	.02785	.02488	.02275	.02125	.02021	.01954	.01921	.01924	.01989	
	.02075	.02281	.02700	.03800	.22060										
20	.26710	.08049	.05107	.03864	.03174	.02736	.02436	.02223	.02069	.01958	.01882	.01835	.01818	.01833	
	.01887	.01989	.02208	.02627	.03718	.21864									
21	.26313	.07943	.05039	.03811	.03126	.02691	.02392	.02177	.02020	.01904	.01821	.01765	.01735	.01730	
	.01754	.01815	.01932	.02144	.02562	.03645	.21662								
22	.25940	.07842	.04976	.03761	.03083	.02650	.02352	.02136	.01977	.01858	.01770	.01707	.01667	.01648	
	.01653	.01684	.01761	.01871	.02045	.02503	.03577	.21511							
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	.01573	.01585	.01622	.01693	.01817	.02032	.02446	.03514	.21352						
24	.25255	.07658	.04860	.03671	.03004	.02577	.02281	.02066	.01904	.01781	.01686	.01615	.01562	.01527	
	.01508	.01507	.01525	.01587	.01841	.01767	.01883	.02398	.03456	.21201					
25	.24939	.07573	.04807	.03630	.02969	.02545	.02250	.02035	.01873	.01748	.01652	.01577	.01521	.01480	
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26	.24639	.07492	.04758	.03591	.02936	.02515	.02221	.02007	.01844	.01718	.01621	.01544	.01484	.01440	
	.01408	.01391	.01388	.01398	.01423	.01471	.01552	.01681	.01898	.02308	.03351	.20925			
27	.24354	.07415	.04708	.03554	.02904	.02486	.02195	.01981	.01818	.01692	.01593	.01514	.01452	.01403	
	.01370	.01347	.01335	.01336	.01349	.01378	.01430	.01512	.01642	.01859	.02268	.03304	.20789		
28	.24082	.07341	.04663	.03519	.02875	.02480	.02170	.01956	.01794	.01667	.01567	.01487	.01424	.01374	
	.01336	.01308	.01291	.01284	.01289	.01306	.01339	.01393	.01476	.01607	.01824	.02231	.03253	.20678	
29	.23823	.07271	.04619	.03486	.02847	.02434	.02148	.01934	.01771	.01645	.01544	.01463	.01398	.01348	
	.01305	.01275	.01254	.01242	.01239	.01247	.01267	.01303	.01358	.01443	.01574	.01791	.02198	.03217	
	.20563														
30	.23574	.07204	.04578	.03454	.02820	.02411	.02124	.01912	.01750	.01623	.01522	.01440	.01374	.01321	
	.01278	.01245	.01221	.01205	.01197	.01198	.01209	.01232	.01269	.01325	.01412	.01544	.01760	.02163	
	.03178	.20453													
31	.23336	.07139	.04538	.03424	.02795	.02388	.02103	.01892	.01731	.01604	.01502	.01420	.01353	.01298	
	.01254	.01219	.01192	.01173	.01160	.01160	.01160	.01175	.01198	.01238	.01298	.01383	.01516	.01731	
	.02132	.03140	.20349												
32	.23108	.07077	.04499	.03395	.02771	.02367	.02083	.01873	.01712	.01586	.01484	.01401	.01333	.01278	
	.01232	.01195	.01170	.01140	.01130	.01120	.01120	.01130	.01140	.01170	.01210	.01289	.01356	.01489	
	.01704	.02103	.03105	.20249											
33	.22889	.07018	.04463	.03387	.02748	.02345	.02064	.01856	.01695	.01568	.01467	.01383	.01315	.01259	
	.01212	.01174	.01140	.01120	.01100	.01090	.01090	.01090	.01100	.01110	.01140	.01183	.01243	.01331	
	.01464	.01678	.02075	.03071	.20153										
34	.22678	.08860	.04427	.03340	.02725	.02327	.02045	.01838	.01678	.01552	.01450	.01367	.01298	.01241	
	.01194	.01150	.01120	.01100	.01080	.01060	.01060	.01060	.01060	.01070	.01090	.01120	.01160	.01219	
	.01308	.01440	.01654	.02049	.03039	.20061									
35	.22475	.08805	.04393	.03315	.02704	.02308	.02029	.01822	.01663	.01537	.01435	.01352	.01283	.01225	
	.01177	.01140	.01100	.01080	.01060	.01040	.01030	.01020	.01020	.01030	.01040	.01060	.01090	.01140	
	.01187	.01286	.01418	.01631	.02024	.03008	.19872								

36	.22279	.05852	.04360	.03290	.02584	.02290	.02013	.01807	.01648	.01523	.01421	.01337	.01268	.01210
	.01160	.01120	.01090	.01060	.01040	.01020	.01000	.00986	.00983	.00965	.01000	.01020	.01040	.01070
	.01110	.01178	.01265	.01387	.01609	.02000	.02979	.19887						
37	.22090	.05800	.04329	.03286	.02584	.02273	.01987	.01792	.01634	.01509	.01407	.01324	.01254	.01196
	.01150	.01110	.01070	.01040	.01020	.00954	.00963	.00973	.00967	.00986	.00961	.00978	.00983	.01020
	.01050	.01090	.01160	.01248	.01377	.01565	.01978	.02951	.19805					
38	.21904	.05750	.04288	.03243	.02545	.02258	.01982	.01778	.01621	.01496	.01394	.01311	.01241	.01183
	.01130	.01080	.01060	.01030	.01000	.00980	.00984	.00952	.00944	.00940	.00940	.00945	.00955	.00972
	.00995	.01030	.01070	.01140	.01228	.01358	.01559	.01986	.02925	.19726				
39	.21731	.05702	.04268	.03221	.02527	.02240	.01986	.01765	.01608	.01484	.01382	.01298	.01229	.01170
	.01120	.01080	.01040	.01010	.00985	.00964	.00947	.00933	.00923	.00917	.00915	.00917	.00923	.00934
	.00952	.00978	.01010	.01080	.01120	.01208	.01340	.01551	.01938	.02889	.19850			
40	.21561	.05655	.04240	.03200	.02508	.02225	.01954	.01752	.01598	.01472	.01371	.01287	.01217	.01160
	.01110	.01070	.01030	.00998	.00971	.00948	.00931	.00916	.00905	.00897	.00893	.00892	.00895	.00902
	.00915	.00933	.00958	.00992	.01040	.01100	.01192	.01323	.01533	.01916	.02874	.19576		
41	.21395	.05610	.04212	.03178	.02492	.02210	.01941	.01740	.01584	.01460	.01360	.01276	.01208	.01150
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	.00883	.00895	.00915	.00941	.00976	.01020	.01080	.01176	.01307	.01518	.01897	.02851	.19505	
42	.21235	.05566	.04185	.03159	.02575	.02195	.01928	.01728	.01573	.01450	.01349	.01266	.01198	.01140
	.01090	.01040	.01010	.00974	.00946	.00923	.00903	.00886	.00873	.00862	.00855	.00850	.00849	.00850
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43	.21080	.05523	.04159	.03139	.02559	.02182	.01915	.01716	.01562	.01439	.01339	.01256	.01188	.01130
	.01080	.01030	.00985	.00963	.00935	.00911	.00890	.00873	.00858	.00847	.00838	.00832	.00829	.00829
	.00831	.00838	.00848	.00862	.00883	.00908	.00945	.00983	.01060	.01150	.01277	.01484	.01882	.02808
	.19369													
44	.20929	.05481	.04133	.03120	.02544	.02189	.01903	.01705	.01552	.01429	.01329	.01245	.01177	.01120
	.01070	.01020	.00985	.00952	.00924	.00900	.00878	.00860	.00845	.00833	.00823	.00816	.00811	.00809
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	.02785	.19304												
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	.01060	.01010	.00976	.00943	.00914	.00889	.00868	.00849	.00833	.00820	.00808	.00801	.00795	.00792
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	.01828	.02765	.19241											
46	.20640	.05402	.04085	.03084	.02514	.02143	.01880	.01884	.01532	.01410	.01311	.01228	.01160	.01100
	.01050	.01010	.00967	.00934	.00905	.00878	.00857	.00838	.00822	.00808	.00797	.00788	.00781	.00778
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	.01441	.01814	.02745	.19180										
47	.20502	.05384	.04061	.03085	.02500	.02131	.01859	.01874	.01522	.01401	.01302	.01220	.01150	.01090
	.01040	.00997	.00958	.00925	.00898	.00870	.00846	.00826	.00812	.00797	.00785	.00775	.00767	.00762
	.00758	.00757	.00758	.00761	.00767	.00776	.00789	.00805	.00827	.00852	.00883	.00941	.01010	.01090
	.01223	.01427	.01789	.02726	.19121									
48	.20367	.05328	.04038	.03049	.02485	.02119	.01859	.01884	.01513	.01393	.01294	.01212	.01140	.01080
	.01030	.00988	.00950	.00917	.00887	.00861	.00838	.00819	.00802	.00787	.00774	.00764	.00755	.00749
	.00744	.00741	.00741	.00742	.00746	.00753	.00763	.00775	.00793	.00815	.00844	.00881	.00929	.00994
	.01080	.01211	.01414	.01784	.02708	.19063								
49	.20236	.05290	.04016	.03033	.02473	.02107	.01848	.01855	.01505	.01384	.01288	.01204	.01130	.01080
	.01030	.00981	.00943	.00909	.00878	.00853	.00830	.00810	.00793	.00777	.00764	.00753	.00744	.00738
	.00731	.00727	.00726	.00726	.00728	.00733	.00740	.00750	.00764	.00781	.00804	.00833	.00870	.00918
	.00982	.01070	.01200	.01402	.01771	.02390	.19007							
50	.20109	.05255	.03994	.03017	.02460	.02085	.01838	.01848	.01498	.01378	.01278	.01197	.01130	.01070
	.01020	.00974	.00935	.00901	.00872	.00845	.00822	.00802	.00784	.00768	.00755	.00743	.00733	.00725
	.00719	.00714	.00712	.00711	.00712	.00715	.00720	.00727	.00738	.00752	.00770	.00793	.00822	.00859
	.00907	.00972	.01060	.01189	.01380	.01757	.02673	.18953						

$K^* = .40000$
$N = 50$

n															
1	1.00000														
2	.55893	.44107													
3	.46873	.18041	.35086												
4	.41883	.14970	.12090	.31055											
5	.38520	.13453	.09760	.09816	.28651										
6	.36025	.12470	.08585	.07581	.08230	.27005									
7	.34065	.11760	.08021	.06674	.06366	.07329	.25784								
8	.32465	.11200	.07552	.06130	.05550	.05582	.06890	.24626							
9	.31121	.10750	.07194	.05752	.05088	.04826	.05036	.06203	.24052						
10	.29970	.10380	.06905	.05469	.04740	.04384	.04317	.04617	.05830	.23404					
11	.28968	.10040	.06655	.05244	.04485	.04085	.03902	.03937	.04295	.05524	.22852				
12	.28060	.09747	.06460	.05069	.04304	.03885	.03624	.03543	.03641	.04036	.05269	.22373			
13	.27269	.09491	.06281	.04902	.04147	.03693	.03421	.03281	.03285	.03404	.03822	.05053	.21952		
14	.26577	.09262	.06123	.04786	.04016	.03554	.03263	.03089	.03014	.03041	.03209	.03841	.04867	.21577	
15	.25930	.09054	.05982	.04647	.03903	.03438	.03136	.02942	.02832	.02800	.02864	.03045	.03467	.04705	
	.21241														
16	.25339	.08864	.05854	.04541	.03804	.03339	.03031	.02824	.02693	.02626	.02625	.02704	.02905	.03353	
	.04582	.20937													
17	.24798	.08690	.05738	.04445	.03718	.03253	.02941	.02726	.02581	.02493	.02457	.02478	.02572	.02784	
	.03238	.04435	.20680												
18	.24294	.08528	.05631	.04359	.03638	.03177	.02863	.02643	.02489	.02388	.02329	.02315	.02353	.02459	
	.02678	.03131	.04321	.20406											
19	.23828	.08380	.05533	.04278	.03567	.03108	.02795	.02571	.02411	.02299	.02226	.02191	.02196	.02245	
	.02360	.02584	.03038	.04218	.20171										
20	.23395	.08241	.05441	.04206	.03502	.03048	.02734	.02808	.02344	.02225	.02143	.02093	.02074	.02091	
	.02150	.02272	.02501	.02855	.04124	.19954									
21	.22988	.08111	.05356	.04138	.03442	.02992	.02678	.02453	.02285	.02162	.02072	.02012	.01979	.01973	
	.02000	.02087	.02194	.02425	.02879	.04038	.19752								
22	.22605	.07988	.05277	.04075	.03387	.02941	.02636	.02403	.02234	.02106	.02012	.01944	.01901	.01881	
	.01886	.01920	.01993	.02125	.02358	.02809	.03959	.19564							
23	.22251	.07874	.05202	.04016	.03336	.02894	.02564	.02358	.02187	.02058	.01959	.01885	.01835	.01806	
	.01795	.01808	.01849	.01927	.02061	.02298	.02746	.03886	.19387						
24	.21913	.07765	.05131	.03961	.03289	.02850	.02543	.02316	.02145	.02014	.01913	.01836	.01779	.01741	
	.01721	.01720	.01739	.01785	.01867	.02004	.02239	.02688	.03818	.19221					
25	.21594	.07663	.05065	.03909	.03244	.02810	.02504	.02279	.02107	.01975	.01871	.01791	.01731	.01687	
	.01660	.01648	.01653	.01678	.01728	.01813	.01952	.02188	.02834	.03755	.19065				
26	.21291	.07565	.05002	.03860	.03202	.02772	.02468	.02244	.02072	.01939	.01834	.01752	.01688	.01640	
	.01607	.01587	.01582	.01593	.01622	.01675	.01763	.01904	.02140	.02563	.03696	.18917			
27	.21004	.07472	.04942	.03813	.03162	.02735	.02435	.02211	.02040	.01906	.01801	.01717	.01650	.01599	
	.01561	.01536	.01523	.01524	.01539	.01572	.01628	.01718	.01880	.02085	.02537	.03641	.18777		
28	.20731	.07384	.04885	.03769	.03125	.02702	.02403	.02181	.02010	.01876	.01770	.01684	.01616	.01563	
	.01521	.01492	.01473	.01465	.01471	.01490	.01528	.01585	.01676	.01819	.02054	.02493	.03589	.18644	
29	.20470	.07300	.04830	.03727	.03089	.02670	.02374	.02153	.01983	.01849	.01742	.01655	.01586	.01530	
	.01486	.01453	.01430	.01417	.01415	.01423	.01445	.01484	.01545	.01637	.01781	.02018	.02452	.03541	
	.18518														
30	.20221	.07219	.04779	.03687	.03056	.02640	.02346	.02128	.01957	.01823	.01715	.01628	.01558	.01500	
	.01454	.01419	.01392	.01375	.01367	.01368	.01380	.01405	.01445	.01508	.01601	.01745	.01980	.02414	
	.03495	.18397													
31	.19983	.07142	.04729	.03649	.03023	.02612	.02319	.02101	.01932	.01798	.01681	.01604	.01532	.01473	
	.01426	.01388	.01369	.01338	.01325	.01321	.01325	.01340	.01387	.01410	.01473	.01588	.01712	.01946	
	.02378	.03451	.18282												
32	.19755	.07066	.04681	.03612	.02993	.02585	.02294	.02077	.01909	.01778	.01669	.01581	.01508	.01449	
	.01400	.01380	.01329	.01305	.01289	.01280	.01279	.01287	.01304	.01333	.01377	.01442	.01537	.01681	
	.01914	.02344	.03410	.18172											
33	.19536	.06987	.04636	.03577	.02963	.02559	.02271	.02055	.01888	.01755	.01647	.01559	.01485	.01428	
	.01376	.01335	.01302	.01276	.01257	.01245	.01239	.01241	.01251	.01270	.01301	.01348	.01412	.01508	
	.01652	.01885	.02312	.03371	.18067										
34	.19326	.06928	.04592	.03544	.02935	.02534	.02248	.02034	.01887	.01735	.01627	.01539	.01466	.01405	
	.01354	.01312	.01277	.01250	.01229	.01214	.01205	.01202	.01206	.01218	.01239	.01271	.01317	.01384	
	.01481	.01625	.01858	.02281	.03333	.17966									
35	.19124	.06863	.04550	.03511	.02908	.02510	.02226	.02013	.01848	.01716	.01609	.01520	.01447	.01385	
	.01334	.01291	.01255	.01226	.01203	.01186	.01174	.01170	.01170	.01174	.01187	.01210	.01243	.01291	
	.01358	.01465	.01598	.01830	.02252	.03298	.17869								

36 .18830 .05785 .04509 .03460 .02882 .02487 .02206 .01994 .01829 .01698 .01591 .01503 .01429 .01367
.01315 .01271 .01235 .01204 .01180 .01160 .01150 .01140 .01130 .01140 .01140 .01150 .01183 .01217
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37 .18743 .05736 .04470 .03450 .02857 .02465 .02185 .01975 .01812 .01681 .01574 .01486 .01412 .01350
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.01193 .01242 .01310 .01408 .01551 .01781 .02199 .03232 .17686

38 .18562 .05679 .04432 .03421 .02833 .02444 .02167 .01958 .01795 .01685 .01558 .01470 .01386 .01334
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39 .18386 .05622 .04385 .03394 .02810 .02424 .02148 .01941 .01778 .01649 .01543 .01455 .01381 .01319
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.01080 .01110 .01150 .01198 .01288 .01385 .01509 .01736 .02150 .03171 .17518

40 .18220 .05567 .04360 .03367 .02788 .02405 .02131 .01924 .01763 .01634 .01528 .01441 .01367 .01305
.01251 .01206 .01170 .01130 .01100 .01080 .01080 .01040 .01030 .01020 .01020 .01020 .01020 .01030
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41 .18057 .05513 .04328 .03340 .02766 .02386 .02114 .01908 .01748 .01620 .01515 .01427 .01354 .01291
.01238 .01192 .01150 .01120 .01090 .01060 .01040 .01030 .01010 .01000 .00988 .00983 .00984 .00989
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42 .17899 .05461 .04293 .03315 .02745 .02368 .02087 .01894 .01734 .01607 .01502 .01414 .01341 .01278
.01225 .01178 .01140 .01100 .01070 .01050 .01030 .01010 .00988 .00984 .00975 .00971 .00969 .00971
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43 .17747 .05411 .04280 .03291 .02725 .02350 .02082 .01879 .01721 .01593 .01489 .01402 .01329 .01268
.01212 .01170 .01130 .01090 .01060 .01030 .01010 .00994 .00979 .00968 .00957 .00950 .00947 .00948
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44 .17599 .05362 .04229 .03267 .02706 .02333 .02066 .01865 .01708 .01581 .01477 .01390 .01317 .01254
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.03039 .17138

45 .17455 .05315 .04199 .03244 .02685 .02317 .02051 .01851 .01695 .01569 .01465 .01379 .01306 .01243
.01189 .01140 .01100 .01070 .01040 .01010 .00986 .00966 .00949 .00935 .00923 .00914 .00908 .00904
.00903 .00905 .00910 .00919 .00931 .00949 .00972 .01000 .01040 .01090 .01160 .01261 .01402 .01624
.02026 .03016 .17070

46 .17316 .05269 .04169 .03222 .02668 .02301 .02037 .01838 .01682 .01557 .01454 .01368 .01295 .01233
.01179 .01130 .01090 .01060 .01020 .00988 .00974 .00964 .00956 .00951 .00949 .00949 .00951 .00956
.00964 .00967 .00982 .00902 .00915 .00933 .00955 .00987 .01030 .01080 .01150 .01248 .01387
.01808 .02008 .02893 .17003

47 .17181 .05224 .04141 .03200 .02650 .02285 .02023 .01825 .01671 .01546 .01443 .01357 .01285 .01222
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48 .17049 .05181 .04113 .03178 .02633 .02270 .02010 .01813 .01659 .01535 .01433 .01347 .01275 .01213
.01160 .01110 .01070 .01040 .01000 .00976 .00952 .00931 .00912 .00898 .00882 .00871 .00862 .00855
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49 .16921 .05139 .04086 .03158 .02616 .02255 .01997 .01801 .01648 .01524 .01423 .01337 .01265 .01203
.01150 .01100 .01060 .01030 .00994 .00966 .00942 .00920 .00901 .00885 .00871 .00858 .00848 .00840
.00835 .00831 .00829 .00829 .00831 .00837 .00844 .00855 .00870 .00890 .00914 .00946 .00986 .01040
.01110 .01206 .01345 .01563 .01958 .02929 .18815

50 .16797 .05097 .04059 .03138 .02599 .02241 .01984 .01790 .01637 .01514 .01413 .01328 .01258 .01194
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.01030 .01100 .01193 .01332 .01549 .01942 .02909 .18755

$K^* = .45000$
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n

1 1.00000

2 .55205 .44795

3 .45295 .19820 .34885

4 .39867 .16285 .13444 .30403

5 .36243 .14495 .10830 .10710 .27726

6 .33578 .13327 .09585 .08462 .09153 .25895

7 .31489 .12470 .08803 .07432 .07119 .08136 .24540

8 .29813 .11800 .08244 .06801 .06205 .06264 .07411 .23461

9 .28406 .11280 .07812 .06358 .05854 .05404 .06623 .06863 .22823

10 .27207 .10800 .07464 .06022 .05273 .04904 .04636 .05157 .06432 .21906

11 .26188 .10400 .07172 .05753 .04987 .04562 .04372 .04411 .04792 .06082 .21300

12 .25252 .10060 .06923 .05530 .04781 .04307 .04057 .03973 .04078 .04497 .05791 .20774

13 .24439 .09753 .06706 .05340 .04574 .04107 .03823 .03677 .03661 .03810 .04253 .05544 .20312

14 .23710 .09481 .06514 .05175 .04416 .03943 .03641 .03459 .03381 .03410 .03585 .04048 .05331 .19901

15 .23080 .09234 .06342 .05030 .04280 .03805 .03492 .03290 .03178 .03142 .03204 .03403 .03872 .05146
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16 .22450 .09010 .06186 .04901 .04161 .03687 .03388 .03153 .03016 .02946 .02948 .03030 .03244 .03719
.04982 .19202

17 .21899 .08805 .06045 .04784 .04055 .03584 .03262 .03039 .02887 .02795 .02758 .02780 .02881 .03106
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18 .21383 .08615 .05916 .04678 .03959 .03492 .03170 .02941 .02781 .02673 .02613 .02599 .02639 .02753
.02985 .03485 .04706 .18623

19 .20924 .08440 .05795 .04581 .03873 .03410 .03088 .02857 .02690 .02573 .02497 .02480 .02484 .02517
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20 .20488 .08277 .05685 .04491 .03793 .03335 .03015 .02782 .02612 .02487 .02401 .02348 .02329 .02347
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21 .20082 .08126 .05582 .04408 .03721 .03268 .02948 .02718 .02543 .02413 .02320 .02256 .02221 .02216
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22 .19702 .07983 .05486 .04331 .03653 .03205 .02889 .02658 .02481 .02349 .02250 .02178 .02133 .02112
.02117 .02154 .02232 .02372 .02620 .03066 .04291 .17708

23 .19345 .07849 .05398 .04259 .03591 .03146 .02834 .02602 .02426 .02291 .02188 .02112 .02058 .02026
.02018 .02030 .02074 .02157 .02300 .02548 .03023 .04207 .17517

24 .19008 .07723 .05311 .04192 .03532 .03095 .02783 .02552 .02376 .02240 .02134 .02054 .01994 .01954
.01933 .01932 .01953 .02002 .02088 .02235 .02484 .02956 .04130 .17338

25 .18682 .07604 .05231 .04128 .03477 .03045 .02736 .02507 .02330 .02193 .02085 .02002 .01938 .01892
.01863 .01851 .01856 .01883 .01936 .02027 .02175 .02425 .02894 .04058 .17169

26 .18392 .07492 .05155 .04068 .03426 .02998 .02693 .02464 .02288 .02151 .02042 .01956 .01889 .01839
.01803 .01783 .01777 .01788 .01820 .01877 .01971 .02120 .02370 .02837 .03890 .17009

27 .18108 .07385 .05083 .04012 .03377 .02954 .02652 .02425 .02250 .02112 .02002 .01914 .01845 .01791
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28 .17838 .07283 .05015 .03956 .03331 .02913 .02613 .02388 .02214 .02078 .01965 .01876 .01805 .01749
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29 .17582 .07186 .04950 .03907 .03288 .02874 .02577 .02354 .02180 .02042 .01932 .01842 .01769 .01711
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30 .17337 .07093 .04868 .03858 .03247 .02837 .02543 .02321 .02149 .02011 .01900 .01810 .01736 .01676
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31 .17104 .07005 .04829 .03811 .03207 .02802 .02511 .02291 .02119 .01982 .01871 .01780 .01706 .01645
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32 .16881 .06920 .04772 .03767 .03169 .02769 .02480 .02262 .02091 .01955 .01844 .01753 .01678 .01616
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33 .16667 .06839 .04718 .03725 .03134 .02737 .02451 .02234 .02065 .01929 .01819 .01727 .01652 .01589
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34 .16462 .06761 .04666 .03684 .03098 .02706 .02423 .02208 .02040 .01905 .01795 .01703 .01627 .01564
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35 .16265 .06687 .04615 .03645 .03068 .02677 .02396 .02183 .02018 .01882 .01772 .01681 .01605 .01541
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.01515 .01618 .01771 .02015 .02457 .03535 .15882

36	.18078	.08615	.04587	.03607	.03034	.02648	.02371	.02159	.01993	.01860	.01750	.01660	.01583	.01519
	.01465	.01419	.01380	.01348	.01323	.01302	.01287	.01278	.01274	.01278	.01285	.01301	.01327	.01363
	.01415	.01487	.01591	.01743	.01986	.02425	.03498	.18782						
37	.15895	.08545	.04521	.03571	.03004	.02622	.02346	.02137	.01972	.01838	.01730	.01639	.01563	.01498
	.01444	.01398	.01358	.01325	.01298	.01278	.01259	.01248	.01241	.01239	.01243	.01254	.01271	.01294
	.01336	.01388	.01481	.01664	.01717	.01958	.02395	.03459	.15686					
38	.15720	.08478	.04478	.03536	.02975	.02597	.02323	.02115	.01951	.01819	.01711	.01620	.01544	.01480
	.01425	.01378	.01338	.01304	.01276	.01252	.01234	.01220	.01211	.01206	.01207	.01212	.01224	.01243
	.01271	.01310	.01363	.01436	.01539	.01691	.01932	.02367	.03424	.15593				
39	.15551	.08414	.04433	.03502	.02948	.02572	.02300	.02094	.01931	.01800	.01692	.01602	.01526	.01461
	.01406	.01358	.01318	.01284	.01255	.01230	.01211	.01195	.01184	.01177	.01174	.01177	.01184	.01197
	.01217	.01248	.01285	.01339	.01412	.01516	.01667	.01907	.02339	.03390	.15504			
40	.15388	.08352	.04381	.03470	.02919	.02548	.02279	.02074	.01913	.01782	.01675	.01585	.01509	.01444
	.01389	.01341	.01300	.01265	.01235	.01210	.01189	.01172	.01160	.01150	.01150	.01140	.01150	.01160
	.01170	.01193	.01222	.01262	.01316	.01390	.01493	.01645	.01883	.02313	.03357	.15418		
41	.15231	.08291	.04361	.03438	.02893	.02525	.02255	.02055	.01894	.01765	.01656	.01566	.01493	.01428
	.01373	.01325	.01283	.01248	.01217	.01191	.01170	.01150	.01140	.01130	.01120	.01120	.01120	.01120
	.01130	.01150	.01170	.01200	.01240	.01295	.01369	.01472	.01623	.01860	.02288	.03326	.15335	
42	.15079	.08233	.04312	.03408	.02867	.02503	.02238	.02036	.01877	.01748	.01642	.01553	.01477	.01413
	.01357	.01309	.01267	.01231	.01200	.01174	.01150	.01130	.01120	.01100	.01100	.01090	.01090	.01090
	.01100	.01110	.01120	.01150	.01178	.01218	.01274	.01349	.01462	.01602	.01838	.02264	.03298	.15255
43	.14932	.08176	.04274	.03379	.02843	.02481	.02218	.02018	.01860	.01732	.01626	.01538	.01462	.01398
	.01342	.01294	.01252	.01216	.01184	.01160	.01130	.01110	.01100	.01080	.01070	.01070	.01060	.01060
	.01070	.01070	.01080	.01100	.01130	.01160	.01200	.01255	.01329	.01433	.01582	.01818	.02241	.03267
	.15177													
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	.01328	.01280	.01238	.01201	.01170	.01140	.01120	.01100	.01080	.01070	.01060	.01050	.01040	.01040
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	.03238	.15101												
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	.01315	.01267	.01224	.01188	.01160	.01130	.01100	.01080	.01080	.01080	.01040	.01030	.01020	.01020
	.01020	.01020	.01020	.01030	.01060	.01080	.01090	.01120	.01180	.01219	.01293	.01398	.01545	.01778
	.02197	.03212	.15028											
46	.14518	.08017	.04167	.03298	.02773	.02420	.02184	.01988	.01813	.01687	.01583	.01498	.01421	.01357
	.01302	.01254	.01211	.01174	.01140	.01110	.01090	.01070	.01050	.01030	.01020	.01010	.01000	.00995
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	.01780	.02177	.03186	.14958										
47	.14388	.05967	.04134	.03270	.02752	.02401	.02147	.01952	.01788	.01673	.01570	.01483	.01408	.01345
	.01290	.01241	.01199	.01160	.01130	.01100	.01080	.01050	.01030	.01020	.01000	.00993	.00984	.00977
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	.01511	.01742	.02157	.03181	.14885									
48	.14262	.05919	.04101	.03244	.02731	.02383	.02130	.01937	.01784	.01680	.01557	.01470	.01397	.01333
	.01278	.01230	.01187	.01150	.01120	.01090	.01080	.01040	.01020	.01000	.00989	.00977	.00984	.00980
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	.01347	.01495	.01726	.02138	.03137	.14822								
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	.01266	.01218	.01176	.01140	.01110	.01080	.01080	.01030	.01010	.00991	.00976	.00983	.00982	.00944
	.00938	.00934	.00932	.00932	.00935	.00940	.00948	.00960	.00978	.00997	.01020	.01060	.01100	.01160
	.01230	.01332	.01478	.01708	.02119	.03114	.14757							
50	.14021	.05825	.04039	.03195	.02690	.02348	.02055	.01908	.01757	.01634	.01533	.01447	.01374	.01310
	.01256	.01207	.01170	.01130	.01090	.01070	.01040	.01020	.00995	.00978	.00983	.00980	.00838	.00829
	.00822	.00917	.00914	.00912	.00914	.00917	.00923	.00932	.00944	.00961	.00982	.01010	.01040	.01090
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36	.13852	.08337	.04551	.03678	.03144	.02778	.02503	.02303	.02140	.02008	.01899	.01807	.01730	.01665
	.01610	.01563	.01524	.01491	.01464	.01443	.01428	.01418	.01414	.01416	.01425	.01442	.01469	.01507
	.01561	.01627	.01744	.01902	.02151	.02600	.03272	.13917						
37	.13478	.08281	.04488	.03638	.03108	.02746	.02480	.02278	.02114	.01983	.01874	.01783	.01707	.01641
	.01586	.01538	.01498	.01464	.01436	.01414	.01396	.01384	.01377	.01375	.01380	.01390	.01408	.01437
	.01478	.01531	.01607	.01714	.01872	.02120	.02584	.03630	.13818					
38	.13311	.08188	.04447	.03585	.03073	.02715	.02452	.02280	.02090	.01959	.01851	.01761	.01884	.01619
	.01563	.01515	.01474	.01439	.01410	.01386	.01367	.01353	.01343	.01338	.01339	.01345	.01358	.01378
	.01407	.01447	.01502	.01579	.01686	.01843	.02080	.02533	.03590	.13719				
39	.13150	.08117	.04388	.03555	.03039	.02685	.02425	.02225	.02065	.01936	.01829	.01739	.01663	.01597
	.01541	.01493	.01452	.01416	.01386	.01361	.01341	.01325	.01313	.01306	.01303	.01306	.01313	.01327
	.01346	.01378	.01419	.01475	.01552	.01659	.01816	.02062	.02502	.03552	.13628			
40	.12985	.08049	.04350	.03516	.03007	.02657	.02399	.02201	.02043	.01915	.01806	.01716	.01642	.01577
	.01521	.01473	.01431	.01395	.01364	.01336	.01316	.01299	.01288	.01277	.01271	.01271	.01274	.01284
	.01298	.01321	.01351	.01393	.01450	.01526	.01634	.01790	.02035	.02472	.03518	.13538		
41	.12845	.07984	.04304	.03481	.02976	.02629	.02374	.02178	.02021	.01894	.01788	.01699	.01623	.01558
	.01502	.01453	.01411	.01375	.01343	.01316	.01294	.01275	.01261	.01250	.01242	.01239	.01240	.01245
	.01255	.01272	.01295	.01326	.01368	.01425	.01502	.01610	.01765	.02009	.02444	.03480	.13448	
42	.12701	.07925	.04260	.03446	.02946	.02603	.02350	.02156	.02000	.01874	.01769	.01680	.01604	.01538
	.01483	.01435	.01392	.01355	.01324	.01296	.01273	.01253	.01237	.01225	.01216	.01211	.01209	.01211
	.01218	.01229	.01248	.01270	.01302	.01345	.01402	.01479	.01587	.01742	.01984	.02417	.03446	.13384
43	.12561	.07859	.04217	.03412	.02917	.02577	.02327	.02134	.01980	.01855	.01750	.01662	.01587	.01522
	.01466	.01417	.01375	.01337	.01306	.01277	.01253	.01233	.01216	.01202	.01192	.01185	.01181	.01181
	.01184	.01192	.01204	.01222	.01247	.01280	.01323	.01380	.01458	.01585	.01719	.01980	.02388	.03414
	.13249													
44	.12430	.07800	.04176	.03378	.02889	.02553	.02305	.02113	.01961	.01838	.01732	.01645	.01570	.01505
	.01449	.01400	.01358	.01320	.01288	.01259	.01235	.01214	.01196	.01181	.01170	.01160	.01160	.01150
	.01150	.01160	.01170	.01181	.01200	.01225	.01258	.01302	.01359	.01437	.01544	.01698	.01936	.02365
	.03382	.13204												
45	.12300	.07742	.04135	.03347	.02862	.02529	.02283	.02093	.01942	.01818	.01715	.01626	.01553	.01489
	.01433	.01384	.01342	.01304	.01271	.01243	.01217	.01196	.01177	.01160	.01150	.01140	.01130	.01130
	.01130	.01130	.01140	.01140	.01160	.01178	.01204	.01238	.01282	.01340	.01417	.01523	.01677	.01916
	.02341	.03352	.13128											
46	.12170	.07667	.04097	.03315	.02836	.02506	.02262	.02074	.01924	.01801	.01698	.01612	.01538	.01474
	.01418	.01369	.01326	.01289	.01256	.01227	.01201	.01179	.01160	.01140	.01130	.01120	.01110	.01110
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	.01895	.02318	.03323	.13054										
47	.12050	.07633	.04058	.03285	.02810	.02483	.02241	.02055	.01906	.01784	.01682	.01596	.01523	.01459
	.01403	.01355	.01312	.01274	.01241	.01211	.01186	.01160	.01140	.01130	.01110	.01100	.01090	.01080
	.01080	.01080	.01080	.01080	.01090	.01100	.01120	.01140	.01160	.01200	.01244	.01302	.01380	.01486
	.01838	.01875	.02295	.03295	.12983									
48	.11930	.07588	.04022	.03256	.02785	.02461	.02223	.02037	.01889	.01768	.01667	.01581	.01508	.01445
	.01389	.01341	.01298	.01260	.01227	.01197	.01170	.01150	.01130	.01110	.01100	.01080	.01070	.01070
	.01060	.01060	.01060	.01060	.01060	.01070	.01080	.01100	.01120	.01160	.01182	.01227	.01285	.01362
	.01468	.01620	.01885	.02273	.03267	.12913								
49	.11810	.07523	.03985	.03225	.02761	.02440	.02203	.02018	.01873	.01752	.01652	.01567	.01494	.01431
	.01376	.01327	.01285	.01247	.01213	.01183	.01160	.01130	.01110	.01100	.01080	.01070	.01060	.01050
	.01040	.01040	.01030	.01030	.01040	.01040	.01050	.01080	.01080	.01100	.01120	.01160	.01210	.01288
	.01345	.01451	.01602	.01838	.02253	.03241	.12846							
50	.11700	.07460	.03952	.03200	.02738	.02419	.02184	.02002	.01857	.01737	.01638	.01553	.01481	.01418
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	.01020	.01020	.01010	.01010	.01010	.01020	.01020	.01030	.01050	.01080	.01090	.01110	.01150	.01194
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8	.23205	.12843	.09585	.08658	.08082	.08148	.09330	.19759							
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 8   .21389  .12993  .10430  .09224  .08694  .08784  .09881  .18838

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	.02023	.01983	.01947	.01916	.01890	.01867	.01849	.01834	.01824	.01817	.01815	.01817	.01825	.01838
	.01856	.01886	.01925	.01977	.02048	.02148	.02285	.02484	.02858	.03634	.07561			
40	.06641	.04607	.03748	.03315	.03023	.02807	.02640	.02504	.02393	.02295	.02218	.02148	.02087	.02035
	.01988	.01947	.01912	.01880	.01853	.01830	.01811	.01795	.01783	.01775	.01770	.01769	.01773	.01782
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	.01742	.01757	.01778	.01806	.01849	.01901	.01972	.02070	.02208	.02417	.02771	.03531	.07381	
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	.01922	.01882	.01848	.01814	.01787	.01763	.01742	.01724	.01710	.01699	.01690	.01686	.01684	.01686
	.01693	.01704	.01720	.01742	.01773	.01813	.01865	.01937	.02034	.02172	.02379	.02730	.03483	.07296
43	.06321	.04295	.03574	.03183	.02885	.02678	.02520	.02390	.02283	.02183	.02115	.02048	.01989	.01937
	.01892	.01851	.01816	.01784	.01756	.01731	.01710	.01692	.01677	.01664	.01655	.01648	.01645	.01645
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46	.06037	.04106	.03419	.03027	.02761	.02565	.02413	.02293	.02186	.02095	.02024	.01959	.01902	.01852
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8	.18236	.13078	.11220	.10280	.09855	.09814	.10840	.16578							
9	.16734	.12010	.10280	.09359	.08850	.08680	.08878	.09857	.15344						
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34	.06340	.04633	.03977	.03591	.03325	.03128	.02970	.02843	.02738	.02649	.02573	.02504	.02452	.02403	.02362 .02327 .02297 .02273 .02254 .02240 .02232 .02230 .02238 .02248 .02267 .02298 .02343 .02406 .02495 .02623 .02820 .03153 .03850 .06920
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28 .06320 .04947 .04382 .04038 .03797 .03614 .03470 .03362 .03266 .03174 .03105 .03048 .03000 .02961
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33 .05554 .04365 .03862 .03581 .03350 .03189 .03061 .02956 .02868 .02793 .02729 .02673 .02626 .02584
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34 .05425 .04256 .03774 .03481 .03274 .03117 .02982 .02889 .02803 .02729 .02665 .02611 .02564 .02523
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35 .05302 .04161 .03691 .03405 .03203 .03049 .02926 .02826 .02741 .02669 .02607 .02553 .02508 .02465
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36	.05188	.04071	.03512	.03332	.03135	.02984	.02884	.02786	.02683	.02612	.02551	.02497	.02451	.02410
	.02375	.02344	.02318	.02298	.02276	.02262	.02251	.02244	.02241	.02243	.02250	.02263	.02284	.02313
	.02353	.02408	.02485	.02585	.02758	.03030	.03577	.05881						
37	.05075	.03885	.03536	.03263	.03070	.02923	.02805	.02709	.02627	.02557	.02497	.02445	.02398	.02359
	.02323	.02292	.02266	.02242	.02223	.02207	.02195	.02186	.02181	.02180	.02184	.02192	.02207	.02228
	.02257	.02298	.02354	.02431	.02539	.02702	.02970	.03509	.05583					
38	.04970	.03803	.03464	.03187	.03006	.02884	.02749	.02654	.02574	.02506	.02447	.02396	.02349	.02309
	.02274	.02243	.02216	.02193	.02173	.02156	.02143	.02133	.02126	.02123	.02123	.02126	.02136	.02153
	.02175	.02206	.02246	.02302	.02379	.02487	.02648	.02912	.03445	.05490				
39	.04868	.03825	.03388	.03134	.02949	.02808	.02695	.02602	.02524	.02455	.02396	.02347	.02302	.02263
	.02228	.02197	.02170	.02146	.02126	.02108	.02094	.02083	.02074	.02069	.02067	.02069	.02075	.02086
	.02102	.02128	.02166	.02197	.02243	.02329	.02437	.02598	.02858	.03384	.05401			
40	.04773	.03751	.03330	.03073	.02882	.02754	.02644	.02552	.02475	.02409	.02352	.02302	.02258	.02218
	.02184	.02153	.02126	.02102	.02081	.02063	.02048	.02036	.02026	.02020	.02018	.02018	.02019	.02026
	.02037	.02054	.02078	.02108	.02151	.02207	.02283	.02389	.02547	.02808	.03225	.05316		
41	.04681	.03678	.03267	.03015	.02836	.02703	.02585	.02505	.02429	.02364	.02308	.02256	.02215	.02178
	.02142	.02111	.02084	.02060	.02039	.02020	.02005	.01992	.01981	.01973	.01968	.01966	.01967	.01971
	.01978	.01991	.02009	.02033	.02065	.02107	.02162	.02238	.02344	.02500	.02756	.03265	.05234	
42	.04593	.03611	.03207	.02981	.02787	.02654	.02547	.02459	.02385	.02321	.02268	.02217	.02174	.02136
	.02101	.02071	.02044	.02020	.01999	.01980	.01964	.01950	.01939	.01930	.01924	.01920	.01919	.01921
	.01928	.01935	.01948	.01966	.01990	.02022	.02064	.02120	.02195	.02300	.02456	.02709	.03215	.05156
43	.04508	.03545	.03149	.02906	.02737	.02607	.02502	.02416	.02343	.02280	.02225	.02177	.02135	.02097
	.02063	.02033	.02006	.01982	.01961	.01942	.01925	.01911	.01899	.01888	.01882	.01877	.01874	.01874
	.01877	.01883	.01892	.01906	.01925	.01950	.01982	.02024	.02080	.02155	.02259	.02413	.02663	.03164
	.05080													
44	.04427	.03462	.03094	.02857	.02688	.02561	.02459	.02374	.02302	.02240	.02187	.02138	.02097	.02060
	.02026	.01997	.01970	.01946	.01924	.01905	.01888	.01874	.01861	.01851	.01843	.01836	.01833	.01831
	.01832	.01836	.01842	.01852	.01867	.01886	.01911	.01944	.01986	.02042	.02116	.02220	.02372	.02619
	.03114	.05004												
45	.04350	.03422	.03040	.02808	.02643	.02518	.02417	.02334	.02263	.02202	.02149	.02103	.02061	.02024
	.01991	.01962	.01935	.01911	.01890	.01870	.01853	.01838	.01826	.01815	.01808	.01799	.01794	.01791
	.01790	.01792	.01796	.01804	.01814	.01829	.01848	.01874	.01907	.01949	.02005	.02078	.02182	.02332
	.02578	.03067	.04838											
46	.04275	.03364	.02969	.02761	.02599	.02478	.02377	.02295	.02225	.02165	.02114	.02068	.02027	.01990
	.01958	.01928	.01902	.01878	.01856	.01837	.01820	.01805	.01792	.01780	.01771	.01763	.01757	.01753
	.01751	.01751	.01754	.01759	.01767	.01778	.01793	.01813	.01839	.01872	.01914	.01970	.02043	.02145
	.02295	.02537	.03021	.04871										
47	.04203	.03306	.02940	.02716	.02557	.02436	.02338	.02258	.02190	.02131	.02078	.02034	.01994	.01958
	.01925	.01896	.01870	.01846	.01825	.01805	.01788	.01773	.01759	.01747	.01737	.01729	.01722	.01717
	.01714	.01713	.01714	.01717	.01723	.01732	.01744	.01759	.01780	.01805	.01836	.01881	.01936	.02009
	.02110	.02251	.02499	.02977	.04808									
48	.04134	.03254	.02893	.02672	.02516	.02397	.02301	.02222	.02155	.02097	.02048	.02002	.01962	.01929
	.01894	.01865	.01839	.01816	.01794	.01775	.01758	.01742	.01728	.01716	.01706	.01697	.01690	.01684
	.01680	.01678	.01677	.01678	.01683	.01689	.01698	.01711	.01727	.01747	.01773	.01806	.01849	.01904
	.01976	.02077	.02224	.02461	.02935	.04744								
49	.04057	.03202	.02847	.02630	.02477	.02360	.02265	.02188	.02121	.02064	.02014	.01970	.01931	.01896
	.01864	.01836	.01810	.01786	.01765	.01746	.01729	.01713	.01699	.01687	.01676	.01666	.01659	.01652
	.01646	.01644	.01643	.01643	.01646	.01650	.01657	.01666	.01678	.01695	.01716	.01742	.01778	.01818
	.01873	.01945	.02045	.02190	.02426	.02894	.04683							
50	.04003	.03152	.02803	.02589	.02439	.02323	.02231	.02154	.02089	.02033	.01984	.01940	.01902	.01867
	.01836	.01807	.01782	.01758	.01737	.01718	.01701	.01685	.01671	.01658	.01647	.01637	.01629	.01622
	.01617	.01613	.01611	.01610	.01611	.01614	.01619	.01626	.01636	.01649	.01665	.01686	.01713	.01748
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6	.19885	.16491	.15187	.14711	.15111	.18814								
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8	.15822	.12950	.11840	.11240	.10950	.11000	.11630	.14767						
9	.14154	.11740	.10710	.10130	.09806	.09687	.09828	.10480	.13457					
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	.06454	.06859												
17	.08311	.06927	.06318	.05941	.05676	.05485	.05332	.05220	.05139	.05089	.05069	.05086	.05150	.05287
	.05556	.06143	.06275											
18	.07923	.06606	.06027	.05667	.05414	.05224	.05079	.04967	.04883	.04825	.04791	.04785	.04812	.04884
	.05025	.05202	.05486	.07830										
19	.07572	.06317	.05764	.05420	.05177	.04994	.04852	.04742	.04656	.04593	.04551	.04531	.04535	.04570
	.04647	.04780	.05065	.05616	.07618									
20	.07253	.06053	.05525	.05186	.04962	.04786	.04648	.04539	.04453	.04387	.04338	.04310	.04300	.04313
	.04384	.04436	.04580	.04842	.05285	.07334								
21	.06963	.05813	.05307	.04991	.04788	.04596	.04482	.04385	.04270	.04202	.04161	.04116	.04088	.04085
	.04114	.04180	.04244	.04389	.04648	.05185	.07075							
22	.06696	.05563	.05108	.04804	.04587	.04423	.04293	.04186	.04103	.04036	.03983	.03943	.03917	.03905
	.03910	.03934	.03984	.04071	.04216	.04472	.04987	.06837						
23	.06451	.05391	.04924	.04631	.04422	.04263	.04137	.04036	.03952	.03884	.03829	.03787	.03757	.03739
	.03734	.03744	.03772	.03828	.03913	.04068	.04310	.04824	.06818					
24	.06225	.05204	.04784	.04472	.04270	.04116	.03994	.03894	.03812	.03745	.03690	.03646	.03613	.03590
	.03578	.03578	.03592	.03624	.03679	.03759	.03913	.04162	.04665	.06418				
25	.06015	.05030	.04666	.04324	.04129	.03980	.03861	.03764	.03684	.03617	.03562	.03517	.03482	.03456
	.03438	.03432	.03437	.03464	.03488	.03546	.03636	.03780	.04025	.04518	.06227			
26	.05821	.04869	.04450	.04187	.03998	.03854	.03738	.03643	.03566	.03499	.03444	.03399	.03362	.03334
	.03314	.03302	.03299	.03307	.03328	.03384	.03423	.03514	.03656	.03898	.04381	.06052		
27	.05638	.04718	.04313	.04059	.03876	.03738	.03624	.03531	.03454	.03390	.03336	.03290	.03252	.03222
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28	.05470	.04578	.04186	.03938	.03762	.03628	.03517	.03427	.03351	.03288	.03234	.03188	.03150	.03119
	.03095	.03077	.03065	.03061	.03065	.03078	.03104	.03144	.03204	.03295	.03436	.03670	.04138	.05735
29	.05311	.04447	.04086	.03827	.03655	.03523	.03417	.03329	.03255	.03193	.03140	.03094	.03056	.03024
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	.05591													
30	.05162	.04323	.03954	.03722	.03556	.03426	.03323	.03237	.03165	.03104	.03051	.03007	.02968	.02935
	.02909	.02888	.02872	.02861	.02856	.02857	.02866	.02883	.02911	.02954	.03018	.03108	.03243	.03471
	.03820	.05458												
31	.05022	.04207	.03866	.03623	.03480	.03335	.03234	.03151	.03080	.03020	.02969	.02926	.02888	.02854
	.02825	.02804	.02788	.02773	.02765	.02763	.02766	.02777	.02796	.02825	.02868	.02930	.03020	.03168
	.03381	.03822	.05328											
32	.04889	.04097	.03748	.03529	.03371	.03248	.03151	.03070	.03001	.02942	.02891	.02847	.02809	.02777
	.02749	.02726	.02707	.02692	.02682	.02677	.02678	.02682	.02694	.02714	.02744	.02788	.02850	.02940
	.03075	.03296	.03730	.05206										
33	.04765	.03993	.03684	.03441	.03267	.03138	.03072	.02983	.02925	.02888	.02818	.02775	.02737	.02704
	.02676	.02652	.02633	.02617	.02605	.02598	.02594	.02598	.02603	.02618	.02638	.02689	.02713	.02776
	.02884	.02998	.03216	.03643	.05094									
34	.04647	.03885	.03665	.03357	.03207	.03083	.02998	.02920	.02854	.02798	.02749	.02706	.02669	.02635
	.02608	.02584	.02564	.02547	.02534	.02524	.02518	.02517	.02521	.02529	.02544	.02566	.02598	.02642
	.02704	.02793	.02925	.03140	.03661	.04985								
35	.04535	.03803	.03480	.03278	.03132	.03018	.02928	.02852	.02787	.02732	.02684	.02641	.02604	.02572
	.02544	.02520	.02498	.02481	.02467	.02456	.02449	.02445	.02446	.02450	.02460	.02478	.02499	.02531
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37 .04328 .03631 .03324 .03131 .02992 .02884 .02797 .02724 .02663 .02609 .02563 .02522 .02488 .02454
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38 .04232 .03551 .03251 .03063 .02927 .02822 .02737 .02668 .02605 .02553 .02507 .02467 .02431 .02400
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	.05500	.05889	.06890	.07381									
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	.05117	.05217	.05404	.05785	.07080								
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	.03785	.03776	.03776	.03787	.03810	.03849	.03913	.04015	.04188	.04525	.05589		
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9	.12030	.11340	.11010	.10810	.10690	.10650	.10700	.10840	.11830					
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	.05394	.05492	.05689	.06281										
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	.05087	.05121	.05218	.05410	.05980									
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	.03815	.03813	.03814	.03821	.03835	.03857	.03891	.03944	.04032	.04196	.04547			
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	.02917	.02908	.02901	.02895	.02890	.02887	.02885	.02887	.02889	.02895	.02902	.02915	.02932	.02955
	.02988	.03037	.03113	.03250	.03618									
34	.03421	.03237	.03145	.03084	.03038	.03002	.02971	.02945	.02923	.02903	.02886	.02871	.02857	.02845
	.02835	.02826	.02818	.02812	.02807	.02804	.02801	.02801	.02802	.02806	.02811	.02820	.02832	.02849
	.02873	.02906	.02964	.03028	.03163	.03522								
35	.03328	.03150	.03061	.03001	.02957	.02921	.02891	.02865	.02844	.02825	.02808	.02793	.02780	.02768
	.02758	.02749	.02741	.02735	.02729	.02726	.02722	.02721	.02721	.02723	.02727	.02733	.02742	.02754
	.02771	.02795	.02827	.02875	.02948	.03060	.03431							

35	.03241	.03067	.02981	.02923	.02879	.02845	.02815	.02791	.02770	.02751	.02735	.02720	.02707	.02695
	.02685	.02676	.02668	.02661	.02656	.02651	.02648	.02646	.02645	.02645	.02645	.02652	.02659	.02668
	.02680	.02694	.02721	.02763	.02820	.02872	.03002	.03246						
37	.03156	.02989	.02904	.02848	.02806	.02772	.02744	.02720	.02699	.02681	.02665	.02651	.02638	.02626
	.02616	.02607	.02599	.02592	.02586	.02582	.02578	.02575	.02574	.02573	.02575	.02577	.02582	.02589
	.02598	.02611	.02624	.02651	.02683	.02729	.02800	.02927	.03264					
38	.03079	.02914	.02832	.02778	.02737	.02704	.02675	.02653	.02633	.02615	.02599	.02585	.02572	.02561
	.02551	.02542	.02534	.02527	.02521	.02516	.02512	.02509	.02506	.02506	.02506	.02507	.02510	.02515
	.02522	.02532	.02545	.02562	.02585	.02617	.02662	.02732	.02857	.03187				
39	.03004	.02844	.02764	.02711	.02671	.02638	.02612	.02589	.02569	.02552	.02538	.02522	.02510	.02499
	.02488	.02480	.02472	.02465	.02459	.02454	.02448	.02446	.02443	.02442	.02441	.02442	.02444	.02447
	.02452	.02459	.02469	.02482	.02498	.02522	.02554	.02598	.02667	.02790	.03114			
40	.02933	.02776	.02689	.02647	.02606	.02576	.02550	.02528	.02509	.02492	.02477	.02463	.02451	.02440
	.02430	.02421	.02413	.02406	.02400	.02395	.02390	.02386	.02383	.02381	.02380	.02380	.02381	.02383
	.02387	.02392	.02400	.02410	.02423	.02440	.02462	.02494	.02538	.02606	.02728	.03044		
41	.02865	.02712	.02637	.02588	.02548	.02517	.02492	.02470	.02451	.02435	.02420	.02406	.02395	.02384
	.02374	.02365	.02357	.02350	.02344	.02338	.02334	.02330	.02326	.02324	.02322	.02322	.02322	.02323
	.02328	.02330	.02338	.02343	.02353	.02366	.02383	.02406	.02437	.02480	.02547	.02666	.02977	
42	.02801	.02651	.02577	.02526	.02491	.02461	.02436	.02415	.02396	.02380	.02366	.02353	.02341	.02330
	.02321	.02312	.02304	.02297	.02291	.02285	.02280	.02276	.02273	.02270	.02268	.02267	.02266	.02267
	.02269	.02271	.02276	.02281	.02289	.02299	.02312	.02329	.02352	.02382	.02426	.02491	.02606	.02913
43	.02739	.02583	.02521	.02472	.02436	.02407	.02383	.02362	.02344	.02328	.02314	.02301	.02290	.02279
	.02270	.02261	.02253	.02246	.02240	.02234	.02229	.02225	.02221	.02218	.02215	.02214	.02214	.02214
	.02215	.02217	.02220	.02224	.02230	.02238	.02248	.02261	.02278	.02300	.02330	.02372	.02438	.02562
	.02863													
44	.02680	.02537	.02452	.02420	.02384	.02356	.02332	.02312	.02294	.02278	.02264	.02252	.02241	.02230
	.02221	.02212	.02205	.02198	.02191	.02185	.02181	.02178	.02173	.02169	.02167	.02165	.02164	.02163
	.02163	.02165	.02167	.02170	.02176	.02181	.02189	.02199	.02212	.02229	.02251	.02281	.02323	.02387
	.02489	.02795												
45	.02824	.02484	.02415	.02369	.02334	.02306	.02283	.02263	.02246	.02231	.02217	.02205	.02194	.02184
	.02174	.02166	.02158	.02151	.02145	.02139	.02134	.02130	.02126	.02123	.02120	.02118	.02116	.02115
	.02115	.02116	.02117	.02119	.02123	.02128	.02134	.02142	.02152	.02165	.02182	.02204	.02233	.02275
	.02338	.02449	.02739											
46	.02570	.02433	.02365	.02320	.02286	.02259	.02238	.02217	.02200	.02185	.02172	.02160	.02149	.02139
	.02130	.02122	.02114	.02107	.02101	.02095	.02090	.02086	.02081	.02078	.02075	.02073	.02071	.02070
	.02068	.02068	.02070	.02072	.02074	.02076	.02083	.02088	.02097	.02107	.02120	.02137	.02159	.02188
	.02229	.02291	.02400	.02586										
47	.02618	.02384	.02318	.02274	.02241	.02214	.02192	.02173	.02158	.02141	.02128	.02116	.02106	.02098
	.02087	.02079	.02071	.02065	.02058	.02053	.02048	.02043	.02039	.02035	.02032	.02030	.02028	.02026
	.02025	.02025	.02025	.02026	.02028	.02031	.02035	.02040	.02046	.02054	.02064	.02077	.02094	.02116
	.02145	.02185	.02248	.02354	.02534									
48	.02466	.02337	.02272	.02229	.02197	.02171	.02149	.02130	.02114	.02100	.02087	.02075	.02064	.02055
	.02046	.02038	.02031	.02024	.02018	.02012	.02007	.02002	.01998	.01995	.01991	.01989	.01986	.01985
	.01983	.01983	.01983	.01983	.01984	.01986	.01989	.01993	.01998	.02005	.02013	.02023	.02036	.02053
	.02074	.02103	.02143	.02203	.02309	.02585								
49	.02420	.02292	.02228	.02186	.02154	.02129	.02108	.02089	.02073	.02059	.02047	.02035	.02025	.02015
	.02007	.01999	.01992	.01985	.01979	.01973	.01968	.01963	.01959	.01955	.01952	.01949	.01947	.01945
	.01943	.01943	.01942	.01942	.01943	.01944	.01947	.01950	.01954	.01959	.01965	.01974	.01984	.01997
	.02013	.02034	.02063	.02102	.02162	.02255	.02538							
50	.02374	.02246	.02186	.02145	.02114	.02089	.02068	.02050	.02034	.02021	.02008	.01997	.01987	.01977
	.01969	.01961	.01954	.01947	.01941	.01936	.01931	.01928	.01922	.01918	.01915	.01912	.01908	.01907
	.01905	.01904	.01903	.01903	.01903	.01904	.01906	.01908	.01912	.01916	.01921	.01928	.01936	.01946
	.01959	.01975	.01996	.02024	.02063	.02122	.02226	.02493						

$K^* = .89000$
$N = 50$

n													
1	1.00000												
2	.50068	.49932											
3	.33484	.33167	.33349										
4	.25182	.24908	.24843	.25068									
5	.20191	.19982	.19872	.19876	.20099								
6	.16868	.16865	.16877	.16843	.16873	.16784							
7	.14474	.14307	.14227	.14185	.14178	.14218	.14414						
8	.12643	.12537	.12480	.12420	.12400	.12410	.12450	.12638					
9	.11280	.11180	.11090	.11050	.11030	.11020	.11030	.11040	.11280				
10	.10170	.10050	.09985	.09957	.09932	.09919	.09917	.09932	.09961	.10140			
11	.09256	.09150	.09098	.09061	.09035	.09021	.09014	.09018	.09034	.09083	.09233		
12	.08493	.08395	.08346	.08313	.08290	.08274	.08264	.08261	.08267	.08288	.08334	.08476	
13	.07846	.07757	.07711	.07681	.07658	.07643	.07632	.07628	.07626	.07633	.07654	.07700	.07834
14	.07292	.07208	.07167	.07138	.07117	.07101	.07090	.07083	.07080	.07082	.07091	.07111	.07158 .07284
15	.06811 .06406	.06734	.06694	.06667	.06648	.06633	.06621	.06613	.06608	.06607	.06611	.06621	.06641 .06685
16	.06369 .06272	.06316	.06281	.06255	.06237	.06222	.06211	.06203	.06197	.06194	.06194	.06199	.06208 .06230
17	.06017 .05887	.05950	.05915	.05892	.05874	.05860	.05849	.05841	.05835	.05831	.05829	.05831	.05836 .05847
18	.05687 .05524	.05623	.05590	.05568	.05551	.05538	.05527	.05519	.05513	.05508	.05505	.05505	.05507 .05513
19	.05391 .05225	.05331	.05300	.05278	.05262	.05250	.05240	.05231	.05225	.05220	.05217	.05215	.05216 .05218
20	.05124 .04959	.05067	.05038	.05017	.05002	.04990	.04980	.04972	.04968	.04961	.04957	.04955	.04954 .04956
21	.04862 .04720	.04829	.04800	.04781	.04767	.04756	.04748	.04738	.04732	.04727	.04723	.04720	.04718 .04718
22	.04653 .04504	.04611	.04585	.04568	.04553	.04542	.04532	.04525	.04519	.04514	.04509	.04508	.04504 .04503
23	.04462 .04307	.04413	.04388	.04370	.04357	.04346	.04338	.04330	.04324	.04319	.04315	.04312	.04309 .04308
24	.04278 .04128	.04231	.04207	.04190	.04178	.04167	.04159	.04152	.04146	.04141	.04137	.04133	.04131 .04129
25	.04108 .03963	.04064	.04041	.04025	.04012	.04003	.03994	.03988	.03982	.03977	.03972	.03969	.03967 .03965
26	.03953 .03811	.03908	.03887	.03872	.03860	.03850	.03843	.03835	.03830	.03826	.03821	.03818	.03815 .03813
27	.03808 .03671	.03768	.03745	.03730	.03718	.03710	.03702	.03695	.03690	.03685	.03681	.03678	.03675 .03673
28	.03673 .03540	.03633	.03612	.03598	.03587	.03579	.03571	.03565	.03560	.03555	.03551	.03548	.03545 .03542
29	.03548 .03418 .03564	.03508	.03489	.03476	.03465	.03457	.03450	.03444	.03438	.03434	.03430	.03427	.03424 .03421
30	.03431 .03306 .03377	.03384	.03374	.03361	.03351	.03343	.03336	.03330	.03325	.03321	.03317	.03314	.03311 .03308
31	.03321 .03200 .03242	.03285	.03267	.03254	.03244	.03236	.03230	.03224	.03219	.03215	.03211	.03208	.03205 .03202
32	.03219 .03101 .03127	.03184	.03166	.03153	.03144	.03136	.03130	.03124	.03120	.03116	.03112	.03109	.03106 .03103
33	.03122 .03008 .03023	.03088	.03071	.03059	.03050	.03042	.03036	.03031	.03025	.03022	.03019	.03015	.03013 .03010
34	.03031 .02920 .02928	.02999	.02982	.02970	.02961	.02954	.02948	.02943	.02938	.02934	.02931	.02928	.02925 .02922
35	.02945 .02838 .02841	.02914	.02897	.02886	.02876	.02871	.02865	.02860	.02855	.02851	.02848	.02845	.02842 .02840

```
36   .02865 .02834 .02818 .02807 .02795 .02793 .02786 .02781 .02777 .02773 .02770 .02767 .02764 .02762
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      .02759 .02762 .02767 .02774 .02784 .02798 .02823 .02864

37   .02768 .02758 .02742 .02732 .02724 .02717 .02712 .02707 .02703 .02699 .02696 .02693 .02690 .02688
      .02686 .02684 .02681 .02681 .02680 .02679 .02678 .02677 .02677 .02677 .02677 .02676 .02678 .02680
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38   .02715 .02686 .02671 .02661 .02653 .02647 .02641 .02637 .02633 .02629 .02626 .02623 .02620 .02618
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39   .02646 .02618 .02603 .02593 .02586 .02580 .02574 .02570 .02566 .02562 .02559 .02556 .02554 .02552
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      .02478 .02479 .02480 .02482 .02485 .02489 .02493 .02500 .02509 .02522 .02546 .02601

41   .02519 .02492 .02478 .02466 .02461 .02455 .02450 .02446 .02442 .02439 .02436 .02433 .02431 .02429
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      .02417 .02417 .02419 .02420 .02422 .02425 .02429 .02433 .02440 .02448 .02462 .02485 .02539

42   .02459 .02433 .02418 .02410 .02403 .02397 .02393 .02388 .02385 .02381 .02378 .02376 .02374 .02371
      .02369 .02366 .02366 .02364 .02363 .02362 .02361 .02360 .02358 .02358 .02358 .02358 .02358 .02358
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43   .02403 .02377 .02364 .02355 .02348 .02342 .02338 .02333 .02330 .02327 .02324 .02321 .02319 .02317
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      .02303 .02304 .02306 .02306 .02307 .02306 .02310 .02313 .02317 .02321 .02326 .02336 .02348 .02371
      .02424

44   .02348 .02324 .02311 .02302 .02295 .02290 .02285 .02281 .02278 .02274 .02272 .02268 .02267 .02265
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      .02251 .02251 .02252 .02252 .02253 .02255 .02256 .02258 .02261 .02265 .02268 .02276 .02284 .02297
      .02318 .02370

45   .02267 .02273 .02260 .02251 .02245 .02239 .02235 .02231 .02228 .02225 .02222 .02219 .02217 .02215
      .02213 .02211 .02210 .02208 .02207 .02206 .02205 .02204 .02203 .02202 .02202 .02201 .02201 .02201
      .02201 .02201 .02201 .02202 .02203 .02204 .02205 .02207 .02208 .02211 .02215 .02218 .02226 .02234
      .02246 .02268 .02318

46   .02248 .02224 .02211 .02203 .02197 .02191 .02187 .02183 .02180 .02177 .02174 .02172 .02169 .02167
      .02166 .02164 .02162 .02161 .02160 .02158 .02157 .02156 .02156 .02155 .02154 .02154 .02153 .02153
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      .02186 .02198 .02218 .02269

47   .02200 .02177 .02165 .02157 .02150 .02145 .02141 .02137 .02134 .02131 .02128 .02125 .02124 .02122
      .02120 .02118 .02117 .02116 .02114 .02113 .02112 .02111 .02110 .02108 .02108 .02108 .02108 .02107
      .02107 .02107 .02107 .02107 .02108 .02108 .02109 .02110 .02112 .02113 .02116 .02118 .02122 .02126
      .02132 .02140 .02152 .02173 .02222

48   .02155 .02132 .02120 .02112 .02106 .02101 .02097 .02093 .02090 .02087 .02085 .02082 .02080 .02078
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      .02063 .02063 .02063 .02063 .02064 .02064 .02065 .02066 .02067 .02068 .02070 .02072 .02075 .02078
      .02082 .02088 .02098 .02108 .02128 .02178

49   .02111 .02088 .02077 .02070 .02064 .02059 .02055 .02051 .02048 .02045 .02043 .02040 .02038 .02036
      .02035 .02033 .02031 .02030 .02029 .02028 .02027 .02026 .02025 .02024 .02023 .02023 .02022 .02022
      .02021 .02021 .02021 .02021 .02021 .02022 .02022 .02023 .02024 .02025 .02026 .02028 .02030 .02033
      .02038 .02040 .02046 .02054 .02066 .02086 .02133

50   .02070 .02048 .02026 .02028 .02023 .02018 .02014 .02010 .02007 .02005 .02002 .02000 .01998 .01995
      .01994 .01993 .01991 .01990 .01989 .01987 .01986 .01985 .01985 .01984 .01983 .01982 .01982 .01981
      .01981 .01981 .01981 .01981 .01981 .01981 .01981 .01982 .01982 .01983 .01984 .01985 .01988 .01990
      .01992 .01995 .02000 .02006 .02014 .02025 .02045 .02091
```

Appendix V: Evaluation of moments of $EV_I(\xi,b)$

By definition, for $p > 0$

$$\phi(p) = \frac{\Gamma'(p)}{\Gamma(p)} \ .$$

But, for any positive integer k

$$\Gamma^{(k)}(p) = \int_0^\infty x^{p-1} \exp(-x)(\ln x)^k dx$$

and

$$\Gamma^{(k)}(1) = \lambda_1 = \int_0^\infty (\ln x)e^{-x} \, dx \ .$$

In general, define $\lambda_i = \int_0^\infty (\ln x)^i e^{-x} \, dx$. In particular $\lambda_1 = \Gamma^{(1)}(1)$ $= -\gamma = -.5772$. From Jahnke (1960) (p. 12)

$$\phi^{(1)}(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}$$

and, by differentiating successively and setting $z = 1$,

$\phi^{(k)}(z) = (-1)^{k-1}k!\zeta(k+1)$, where $\zeta(x)$ is Riemann's zeta function.

Using values of $\zeta(k)$ [Jahnke (1960, p. 37)], we have

$$\phi^{(1)}(1) = \frac{\pi^2}{6}$$

$$\phi^{(2)}(1) = -2.404$$

$$\phi^{(3)}(1) = \frac{\pi^4}{15} \quad .$$

Under the homogeneous model, we obtain

$$\mu_2 = E[\ell n\ X - E(\ell n\ X)]^2 = b^2(\lambda_2 - \lambda_1^2)$$

$$\mu_3 = b^3(\lambda_3 - 3\lambda_2\lambda_1 + 2\lambda_1^3) = -2.4036b^3$$

$$\mu_4 = b^4(\lambda_4 - 4\lambda_3\lambda_1 + 6\lambda_2\lambda_1^2 - 3\lambda_1^4) = 14.6119b^4$$

(see [Menon (1963)]. Here

$$\lambda_2 = \gamma^2 + \frac{\pi^2}{6}, \quad \lambda_3 = -5.4445, \quad \lambda_4 = 235601.$$

However, under the exchangeable model, using $Y = \ell n\ X$,

$$E_{het}(Y) = \frac{b\gamma}{n}\{n - 1 + \frac{1}{k^*}\}$$

$$E_{het}(Y^2) = \frac{\pi^2 b^2}{6n}\{n - 1 + \frac{1}{k^{*2}}\} + \{\frac{b\gamma}{n}\ (n - 1 + \frac{1}{k^*})\}^2$$

$$E_{het}(Y^3) = \frac{n-1}{n}(-b)^3\phi^{(2)}(1) + \frac{1}{n}\left(\frac{-b}{k^*}\right)^3\phi^{(2)}(1)$$

$$= \frac{\phi^{(2)}(1)(-b^3)}{n}\left\{n - 1 + \frac{1}{k^{*3}}\right\}$$

$$E_{het}(Y^4) = \frac{n-1}{n}(b^4)\phi^{(3)}(1) + \frac{1}{n}\left(\frac{-b}{k^*}\right)^4\phi^{(3)}(1)$$

$$= \frac{\phi^{(3)}(1)(b^4)}{n}\left\{n - 1 + \frac{1}{k^{*4}}\right\}$$

where $E_{\text{homogeneous}}(Y^r) = -b^r\phi^{(r)}(1)$ [see Patel et al (1976)].

Appendix VI Computer Programs

Program A was used to compute $u(r,n,k^*)$ for the Weibull distribution using the recursive formula from Chapter V.3, p. 103. Extended precision Fortran was used to perform the calculations.

Program A

```
      IMPLICIT LOGICAL*1 (A-Z)
      DIMENSION U(50,50), EFUN(501), UBAR(50)
      REAL*16 U, EFUN, UBAR, SUM, K, C, Y, YINCR
      INTEGER I, J, N, R
C
      WRITE(6,200)
 200  FORMAT(' Enter K (with decimal point) and N
     1       ' (integer)'/ each followed by a comma.')
      READ(5,100) K, N
 100  FORMAT(F7.5,I2)
      WRITE(7,201) K, N
 201  FORMAT('1K = ',F7.5,/' N = ',I2,/'0')
C
C INITIALIZATIONS
C
      DO 300 I=1,N
        DO 301 J=1,N
          U(J,I) = 0.0Q0
 301    CONTINUE
 300  CONTINUE
C
      YINCR = (1.Q0/500.Q0)
  DO 302 I=1,501
    Y = QFLOAT(I-1)*YINCR
    IF (Y .GT. 1.Q-30) GO TO 3021
    EFUN(I) = 0.0Q0
    GO TO 302
1   EFUN(I) = QEXP(-(QABS(QLOG(1.Q0/Y)))**(1.Q0/K))
  CONTINUE
CALCULATE VALUES OF U-BAR (FIRST COLUMN OF U).
USE SIMPSON'S (1/3) RULE FOR THE INTEGRATION.
  DO 310 R=1,N
    IF (EFUN(1) .LT. 1.Q-70) EFUN(1) = 1.Q-70
    SUM = EFUN(1)**(R-1)
    DO 311 J=1,249
      IF (EFUN(2*J) .LT. 1.Q-70) EFUN(2*J) = 1.Q-70
      IF (EFUN(2*J+1) .LT. 1.Q-70) EFUN(2*J+1) = 1.Q-70
      SUM = SUM + 4.Q0*EFUN(2*J)**(R-1)
1             + 2.Q0*EFUN(2*J+1)**(R-1)
311    CONTINUE
       SUM = SUM + 4.Q0*EFUN(500)**(R-1) + EFUN(501)**(R-1)
       UBAR(R) = SUM*(.002Q0/3.Q0)
310  CONTINUE
C
C SET 1ST COLUMN OF U EQUAL TO UBAR
C
      DO 315 I=1,N
        U(I,1) = UBAR(I)
315  CONTINUE
```

```
      WRITE(7,250) U(1,1)
  250 FORMAT(14('0',F8.5)/,3(3X,14(1X,F8.5)/)/)
C
C  CALCULATE THE REST OF U AND WRITE IT OUT.
C
      DO 320 I=2,N
        DO 321 R=2,I
          SUM = 0.Q0
          DO 322 J=1,R
            CALL COMB(R-1,J-1,C)
            SUM = SUM + C*((-1)**(J-1))*UBAR(I-R+J)
  322     CONTINUE
          CALL COMB(I-1,R-1,C)
          U(I,R) = C*SUM
  321   CONTINUE
        WRITE(7,250) (U(I,R),R=1,I)
  320 CONTINUE
      STOP
      END
      SUBROUTINE COMB(I,J,C)
      IMPLICIT REAL*16 (A-H,K,O-Z)
      REAL*16 K(15)
C
      DO 300 IK=1,15
        K(IK) = 0.Q0
  300 CONTINUE
C
      LSTOP = MIN0(I-J,J)
      L=1
      PN=QFLOAT(L)
      PD=QFLOAT(L)
      IF (LSTOP .LE. 0) GO TO 3999
      DO 3000 L=1,LSTOP
        A=QFLOAT(L)
        PD=A*PD
        M=I-L+1
        A=QFLOAT(M)
        PN=A*PN
        DO 3001 IK=1,15
          K(IK) = K(IK)+1
 3001   CONTINUE
        IF (K(1) .NE. 2.Q0) GO TO 3101
          FACT=2.Q0
          PN=PN/FACT
          PD=PD/FACT
          K(1) = 0.Q0
 3101   CONTINUE
        IF (K(2) .NE. 3.Q0) GO TO 3102
          FACT=3.Q0
          PN=PN/FACT
          PD=PD/FACT
          K(2) = 0.Q0
```

```
3102   CONTINUE
       IF (K(3) .NE. 5.Q0) GO TO 3103
         FACT=5.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(3) = 0.Q0
3103   CONTINUE
       IF (K(4) .NE. 7.Q0) GO TO 3104
         FACT=7.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(4) = 0.Q0
3104   CONTINUE
       IF (K(5) .NE. 11.Q0) GO TO 3105
         FACT=11.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(5) = 0.Q0
3105   CONTINUE
       IF (K(6) .NE. 13.Q0) GO TO 3106
         FACT=13.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(6) = 0.Q0
3106   CONTINUE
       IF (K(7) .NE. 17.Q0) GO TO 3107
         FACT=17.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(7) = 0.Q0
3107   CONTINUE
       IF (K(8) .NE. 19.Q0) GO TO 3108
         FACT=19.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(8) = 0.Q0
3108   CONTINUE
       IF (K(9) .NE. 23.Q0) GO TO 3109
         FACT=23.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(9) = 0.Q0
3109   CONTINUE
       IF (K(10) .NE. 29.Q0) GO TO 3110
         FACT=29.Q0
         PN=PN/FACT
         PD=PD/FACT
         K(10) = 0.Q0
```

```
3110  CONTINUE
      IF (K(11) .NE. 31.Q0) GO TO 3111
        FACT=31.Q0
        PN=PN/FACT
        PD=PD/FACT
        K(11) = 0.Q0
3111  CONTINUE
      IF (K(12) .NE. 37.Q0) GO TO 3112
        FACT=37.Q0
        PN=PN/FACT
        PD=PD/FACT
        K(12) = 0.Q0
3112  CONTINUE
      IF (K(13) .NE. 41.Q0) GO TO 3113
        FACT=41.Q0
        PN=PN/FACT
        PD=PD/FACT
        K(13) = 0.Q0
3113  CONTINUE
      IF (K(14) .NE. 43.Q0) GO TO 3114
        FACT=43.Q0
        PN=PN/FACT
        PD=PD/FACT
        K(14) = 0.Q0
3114  CONTINUE
      IF (K(15) .NE. 47.Q0) GO TO 3115
        FACT=47.Q0
        PN=PN/FACT
        PD=PD/FACT
        K(15) = 0.Q0
3115  CONTINUE
3000 CONTINUE
3999 CONTINUE
     C=PN/PD
     RETURN
     END
```

Program B was used to generate 25 random samples each of size 5 with one outlier, starting from a uniform (0,1) distribution. The samples of Weibulls are printed out as X's and transformed to W's, samples of EV_I, by a ℓn transformation. For each sample $\bar{w}$ and s_w^2 are computed. Also for each sample the smallest observation is replaced by the second smallest and by the largest and the new means $\bar{w}_{(2,1)}$, $\bar{w}_{(n,1)}$ and variances $s^2_{(2,1)}$, $s^2_{(n,1)}$ are computed. Also the largest observation is replaced by the second largest observation and by the smallest observation and again means $\bar{w}_{(n-1,n)}$, $\bar{w}_{(1,n)}$ and variances $s^2_{(n-1,n)}$, $s^2_{(1,n)}$ are computed for each sample.

Program B

```
C     This program computes Weibull and EV variables,
C     every fifth one spurious.
C     It prints out samples of size 5
C     along with mean and variance.
C     For each sample it also computes
C     the Winsorized mean and variance
C     by replacing the smallest
C     observation or the largest observation.
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION  X(125), W(125), W21(125), W51(125),
       W45(125), W15(125)
      DIMENSION WB(25), WV(25), W21B(25), W21V(25),
        W51B(25), W51V(25)
      DIMENSION W45B(25), W45V(25), W15B(25), W15V(25)
      REAL RAN(125)
      INTEGER ISEED(2)
      INTEGER*2 HSEED(3)
      REAL*8 DSEED
      EQUIVALENCE (ISEED(2),HSEED(1))
C
      WRITE(6,200)
 200  FORMAT(' Enter BETA and K')
      READ(5,100) BETA, C
 100  FORMAT(2F12.5)
      WRITE(7,201) BETA, C
 201  FORMAT('1      BETA = ',F12.5,'    K = ',F7.5)
      BETAC = BETA*C
      NSIZE = 5
      NSAMP = 25
      NTOTAL = NSIZE*NSAMP
C
C Get random number depending on time of day into DSEED
C
      CALL TIME(4,0,ISEED)
      HSEED(3) = HSEED(1)
      HSEED(1) = HSEED(2)
      HSEED(2) = HSEED(3)
      DSEED = DABS(DFLOAT(ISEED(2)))
C
C Call *IMSL routine GGUBS to get NTOTAL random
C   numbers into RAN
C
      NR = NTOTAL
      CALL GGUBS(DSEED, NR, RAN)
C
```

```
C Compute Weibull r.v.'s X and W
C
      DO 301 K=1,NTOTAL
        FACT = BETA
        IF (MOD(K,5) .EQ. 0)   FACT = BETAC
        X(K) = DBLE(-ALOG(RAN(K)))**(1.D0/FACT)
        W(K) = DLOG(X(K))
 301  CONTINUE
C
C Print out X and W
C
      WRITE(7,202)
 202  FORMAT('-            X'/)
      WRITE(7,203) X
 203  FORMAT(25(3X,5(1X,E13.6)/))
      WRITE(7,204)
 204  FORMAT('0            W')
      WRITE(7,203) W
C
C Compute Means and Variances for each sample of size
C NSIZE in X and W
C
      K = 1
      DO 310 KK=1,NSAMP
        CALL STAT(W, K, NSIZE, WB(KK), WV(KK))
        K = K+NSIZE
 310  CONTINUE
C
C Find the smallest, second smallest, largest, and second
C largest entry in each of the samples in W.
C
      K = 1
      DO 320 KK=1,NSAMP
        WMIN1 = 1.D75
        WMAX1 = -1.D75
        DO 321 K1=1,NSIZE
          IF (W(K) .GE. WMIN1)  GO TO 3211
            WMIN2 = WMIN1
            WMIN1 = W(K)
            KMIN = K
            GO TO 3212
 3211     CONTINUE
          IF (W(K) .GE. WMIN2)  GO TO 3212
            WMIN2 = W(K)
 3212     CONTINUE
```

```
C
          IF (W(K) .LE. WMAX1)  GO TO 3216
            WMAX2 = WMAX1
            WMAX1 = W(K)
            KMAX = K
            GO TO 3217
 3216     CONTINUE
          IF (W(K) .LE. WMAX2)  GO TO 3217
            WMAX2 = W(K)
 3217     CONTINUE
          K = K + 1
 321    CONTINUE
C
C Replace the smallest and largest entries with
C the second smallest
C and largest and with the second largest and
C smallest entries to
C form the altered samples W(21), W(51), W(45), and W(15).
C Recompute the mean and variance for each of
C the altered samples.
C
        K = K - NSIZE
        DO 322 K1=1,NSIZE
          W21(K) = W(K)
          IF (K .EQ. KMIN)  W21(K) = WMIN2
          W51(K) = W(K)
          IF (K .EQ. KMIN)  W51(K) = WMAX1
          W45(K) = W(K)
          IF (K .EQ. KMAX)  W45(K) = WMAX2
          W15(K) = W(K)
          IF (K .EQ. KMAX)  W15(K) = WMIN1
          K = K + 1
 322    CONTINUE
 320  CONTINUE
C
      K = 1
      DO 330 KK=1,NSAMP
        CALL STAT(W21, K, NSIZE, W21B(KK), W21V(KK))
        CALL STAT(W51, K, NSIZE, W51B(KK), W51V(KK))
        CALL STAT(W45, K, NSIZE, W45B(KK), W45V(KK))
        CALL STAT(W15, K, NSIZE, W15B(KK), W15V(KK))
        K = K + NSIZE
 330  CONTINUE
      WRITE(7,225)
```

```
 225  FORMAT('$**$FORMAT=FMTL1 FONTNEXTIMAGE=
        1200.MEDIUM.9.FIXED.LANDSCAPE.1')
      WRITE(7,250)
 250  FORMAT('1 Sample      WBAR
               WVAR       W(21)BAR W(21)VAR    W(51)
     BAR W(51)VAR    W(45)BAR W(45)VAR
      W(15)BAR W(15)VAR
      DO 350 J=1,NSAMP
C
C Reassign these so the WRITE statement will
C fit on one line.
C
        W1 = WB(J)
        W2 = WV(J)
        W3 = W21B(J)
        W4 = W21V(J)
        W5 = W51B(J)
        W6 = W51V(J)
        W7 = W45B(J)
        W8 = W45V(J)
        W9 = W15B(J)
        W10 = W15V(J)
        WRITE(7,255) J, W1, W2, W3, W4, W5, W6,
            W7, W8, W9, W10
 255    FORMAT(6X,I2,2X,5('  ',2(F8.5,1X)),'')
 350  CONTINUE
      STOP
      END
      SUBROUTINE STAT(X, KK, NSIZE, XBAR, XVAR)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(125)
      KSTOP = KK + NSIZE - 1
      SUM = 0.0D0
      SSQ = 0.0D0
      DO 300 K=KK,KSTOP
        SUM = SUM + X(K)
        SSQ = SSQ + (X(K)**2)
 300  CONTINUE
      XBAR = SUM/DFLOAT(NSIZE)
      XVAR = (SSQ - ((SUM**2)/DFLOAT(NSIZE)))/
             (DFLOAT(NSIZE) - 1
      RETURN
      END
```